CS4102 Algorithms

Spring 2020

Warm up:

Show that $P = NP$

Just Kidding!

But you'd get an A and not have to take the final if you could. ☺

Also, you'd be famous, and you'd win $1M US for solving one of the Millennium Prize problems.

Today’s Keywords

- P vs NP
- NP Hard, NP Completeness
- 3SAT
- k-Independent Set
- k-Vertex Cover
- k-Clique

- Readings: CLRS Chapter 34

Summary of Where We Are

- Focusing on “hard” problems, those that seem to be exponential
- Reductions used to show “hardness” relationships between problems
- Starting to define “classes” of problems based on complexity issues
  - P are problems that can be solved in polynomial time
  - NP are problems where a solution can be verified in polynomial time
  - NP-hard are problems that are at least as hard as anything in NP
  - NP-complete are NP-hard problems that “stand or fall together”

Review: P And NP Summary

- **P** = set of problems that can be solved in polynomial time
- **NP** = set of problems for which a solution can be verified in polynomial time
  - Note: this is a more "informal" definition, but it's fine for CS4102
  - See later slide for more info.
- **P** $\subseteq$ **NP**
- Open question: Does **P = NP**?

Review: Reduction

- A problem A can be *reduced* to another problem B if
  - any instance of A can be "rephrased" to an instance of B, such that…
  - the solution to which provides a solution to the instance of A
  - (We sometimes call this rephrasing a *transformation)*
- Intuitively: If A reduces *in polynomial time* to B, A is "no harder to solve" than B
  - I.e. if B is polynomial, A is not exponential

Review:
NP-Hard and NP-Complete

- If A is *polynomial-time reducible* to B, we denote this A $\leq_p$ B

- **Definition of NP-Hard and NP-Complete:**
  - If all problems A $\in$ **NP** are reducible to B , then B is *NP-Hard*
  - We say B is *NP-Complete* if B is NP-Hard and B $\in$ **NP**
- If B $\leq_p$ C and B is NP-Complete, C is also NP-Complete
  - Can you explain why? (Assume C is in NP.)

Before We Move On…

- **Where we want to go next:**
  Are there any NP-Hard problems? Any NP-C problems? How do we show a given problem is NP-C?

- But first, a moment to comment on some “subtleties”
  – Pseudo-polynomial
  – The “real” (i.e. formal) definition of NP

Encodings, Input Sizes

- Our CLRS text takes a formal CS approach to these topics
  - Formal languages, encodings, etc.
  - pp. 1055-1061
- Here in CS4102 we're OK being less formal
  - So we've tried to "translate" or simplify a few things
  - So we're talking about this without those using approach shown on those pages!
- But one point about on encoding and input size…

Subtlety #1: Input Size and P

- **Sometimes a problem seems to be in P but really isn't**
- Example: finding if value n is a prime
  - Just loop and do a mod. That’s just Θ(n), isn't it?
- Note that here “n” is not the count or number of data items.
  - There's just one input item.
  - But “n” is a value with a size that affects the execution time.
  - The size is the number of bits, which is log(n)
  - T(size) = n but size is log(n).
  - T(size) = T(log n) = n = $10^{\log n}$ = 10^{size} This is really an exponential algorithm!
- Be careful when “n” is not a count of data items but a value to be processed
  - E.g. Dynamic programming problems (e.g. a table's dimension)
  - We talked about *pseudo-polynomial run-times* in our lectures on DP

Subtlety #2: NP and Decision Problems

- We've said all this theory is based on reasoning about decision problems
- We've said NP are problems where you can check a solution in polynomial time
- But a <u>solution</u> to a decision problem is yes/no or true/false
  - E.g. k-vertex-cover(G, k) → yes/no
  - If we're only given "yes", how can we verify that solution against G and k without solving the problem?

A Glimpse of Formal Definition of NP

- **NP** (***n****ondeterministic* ***p****olynomial time*) is the set of decision problems that can be solved in polynomial time by a *nondeterministic* computer
  - Think of a non-deterministic computer as a computer that magically "guesses" a what we've thought of as solution, then verifies it is correct
  - Solution for original decision problem is yes/no, but this ***witness*** or ***certificate*** is information that allows us to say "yes" or "no"
  - If answer should be "yes" for an input, computer always guesses right witness
  - Or, you can think of it as a parallel machine that generates all possible witnesses and says "yes" is any of those verifies
- For more, see Wikipedia or other source about these topics

What's Next?

- **Where we want to go next:**
  - Are there any NP-Hard problems? Are there any NP-C problems?
  - How do we show a given problem is NP-C?

- Reminder: why do we care?
  - We know $\boldsymbol{P} \subseteq \boldsymbol{NP}$
  - But are they equal or is it a proper subset?
  - In other words, is there a problem in **NP** that cannot be directly solved in polynomial time?
    Do some problems in NP have an exponential lower bound?
  - Is P = NP? Or not? (The big question!)

Reminder (again)

- **Definition of NP-Hard and NP-Complete:**
  - If all problems A $\in$ **NP** are reducible to B , then B is *NP-Hard*
  - We say B is *NP-Complete* if:
    - B is NP-Hard
    - and B $\in$ **NP**

- If B $\leq_p$ C and B is NP-Complete, C is also NP-Complete
  - We'll see why in two more slides
  - As long as C $\in$ **NP**. Otherwise C $\in$ **NP-hard.**

Proving NP-Completeness

- *What steps do we have to take to prove a problem B is NP-Complete?*
  - Pick a known NP-Hard (or NP-Complete) problem A
    - Assuming there is one! (More later.)
  - Reduce A to B
    - Describe a transformation that maps instances of A to instances of B, such that "yes" for instance of B = "yes" for instance of A
    - Prove the transformation works
    - Prove it runs in polynomial time
  - Oh yeah, prove B $\in$ **NP**

Order of the Reduction When Proving NP-Completeness

- To prove B is **NP-c**, show $A \leq_p B$ where $A \in$ **NP-Hard**
  - Why have the known NP-Hard problem "on the left"? Shouldn't it be the other way around? (No!)
- If $A \in$ **NP-Hard**, then: all NP problems $\leq_p A$
- If you show $A \leq_p B$, then:
  any-NP-problem $\leq_p A \leq_p B$

- Thus any problem in NP can be reduced to B if the two transformations are applied in sequence
  - And both are polynomial
- NP-c are "complete" because: if $A \in$ NP-c and $A \leq_p B$, then $B \in$ NP-c
  - As long as both $\in$ **NP**

“Consequences” of NP-Completeness

- NP-Complete is the set of “hardest” problems in NP, with these important properties:
  - If any *one* NP-Complete problem can be solved in polynomial time…
  - …then *every* NP-Complete problem can be solved in polynomial time…
  - …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
  - Or, prove an exponential lower-bound for *any single* **NP-C** problem, then *every* **NP-C** problem is exponential

Therefore: solve (say) traveling salesperson problem in O(n^{100}) time, you've proved that **P = NP**. Retire rich & famous!

Can a Problem be NP-Hard but not NP-C?

- So, find a reduction and then try to prove B ∈ **NP**
  - *What if you can't?*
- Are there any problems B that are NP-hard but not NP-complete? This means:
  - All problems in NP reduce to B . (A known NP-Hard problem can be reduced to B.)
  - But, B cannot be proved to be in NP
- Yes! Some examples:
  - Non-decision forms of known NP-Cs (e.g. TSP)
  - The halting problem. (Transform a SAT expression to a Turing machine.)
  - Others.

But You Need One NP-Hard First…

- If you have one NP-Hard problem, you can use the technique just described to prove other problems are NP-Hard and NP-c
  - We need an NP-Hard problem to start this off
- The definition of NP-Hard was created to prove a point
  - There *might be* problems that are at least as hard as "anything" (i.e. all NP problems)
- Are there really NP-complete problems?
- **Cook-Levin Theorem: The satisfiability problem (SAT) is NP-Complete.**
    - Stephen Cook proved this "directly", from first principles, in 1971
    - Proven independently by Leonid Levin (USSR)
    - Showed that any problem that meets the definition of NP can be transformed in polynomial time to a CNF formula.
    - Proof outside the scope of this course (lucky you)

More About The SAT Problem

- The first problem to be proved NP-Complete was *satisfiability* (SAT):
  - Given a Boolean expression on *n* variables, can we assign values such that the expression is TRUE?
  - Ex: $((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$
- You might imagine that lots of decision problems could be expressed as a complex logical expression
  - And Cook and Levin proved you were right!
  - Proved the general result that any NP problem can be expressed this way

Conjunctive Normal Form (CNF)

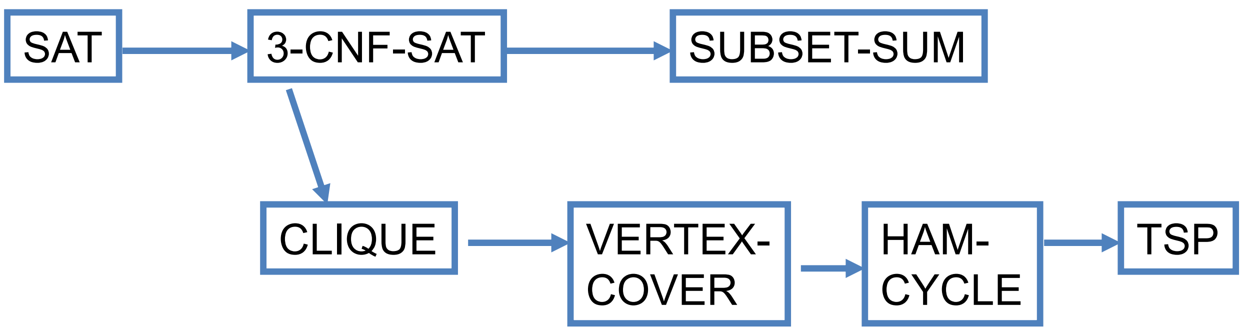
- Even if the form of the Boolean expression is simplified, the problem may be NP-Complete
  - *Literal*: an occurrence of a Boolean or its negation
  - A Boolean formula is in *conjunctive normal form*, or *CNF*, if it is an AND of clauses, each of which is an OR of literals
    - Ex: $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_5)$
  - *3-CNF*: each clause has exactly 3 distinct literals
    - Ex: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_5 \vee x_3 \vee x_4)$
    - Notice: true if at least one literal in each clause is true
  - Note: Arbitrary SAT expressions can be translated into CNF forms by introducing intermediate variables etc.

The 3-CNF Problem

- Satisfiability of Boolean formulas in 3-CNF form (the *3-CNF Problem*) is NP-Complete
  - Proof: Also done by Cook ("part 2" of Cook's theorem)
  - But it's not that hard to show SAT $\leq_p$ 3-CNF
- The reason we care about the 3-CNF problem is that it is relatively easy to reduce to others
  - Thus by proving 3-CNF is NP-Complete we can prove many seemingly unrelated problems are NP-Complete

Joining the Club

- Given one NP-c problem, others can join the club
  - Prove that SAT reduces to another problem, and so on…



  - Membership in NP-c grows…
  - Classic textbook: Garey, M. and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness,* 1979.

Reductions to Prove NP-C

- Next:
  - A tour of how to prove some problems are NP-C
  - 3-SAT is a good starting point!

  - k-Independent Set
  - k-Vertex Cover
  - k-Clique

Reminder about 3-SAT

- Shown to be NP-hard by Cook
- Given a 3-CNF formula (logical AND of clauses, each an OR of 3 variables), is there an assignment of true/false to each variable to make the formula true (i.e., <u>satisfy</u> the formula)?

$$(x \lor y \lor z) \land (x \lor \bar{y} \lor y) \land (u \lor y \lor \bar{z}) \land (z \lor \bar{x} \lor u) \land (\bar{x} \lor \bar{y} \lor \bar{z})$$

Clause

Variables

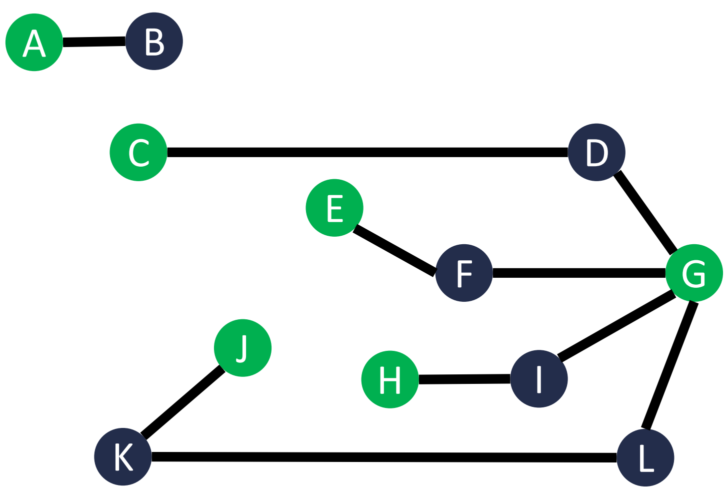
$x = \text{true}$
$y = \text{false}$
$z = \text{false}$
$u = \text{true}$

k-Independent Set is NP-Complete

1. Show that it belongs to NP
2. Show it is NP-Hard
   – Show 3-SAT $\leq_p$ k-Independent Set

k-Independent Set is in NP

- **Show:** For any graph G:
  - There is a short **witness** ("solution" for search problem) that G has a k-independent set
  - The witness can be checked efficiently (in polynomial time)



Graph G

Witness for G: $S = \{A, C, E, G, H, J\}$
(nodes in the k-independent set)

Checking the witness:

- Check that $|S| = k$ $\quad O(k) = O(|V|)$
- Check that every edge is incident on at most one node in S $\quad O(|V| + |E|)$

Total time: $O(|E| + |V|) = \text{poly}(|V| + |E|)$

k-Independent Set is NP-Complete

1. Show that it belongs to NP ✅
2. Show it is NP-Hard
   - Show 3-SAT $\leq_p$ k-Independent Set

3-SAT $\leq_p$ $\boldsymbol{k}$-Independent Set

3-SAT

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z})$

$x = \text{true}$
$y = \text{false}$
$z = \text{false}$
$u = \text{true}$

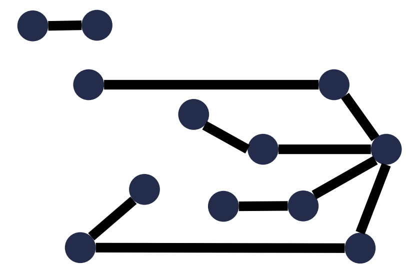
Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

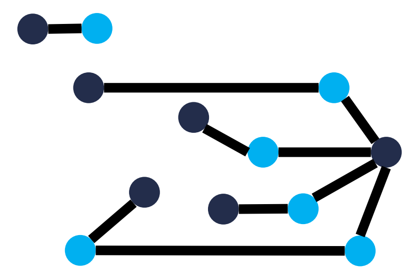
polynomial time

Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

polynomial time

k-independent set

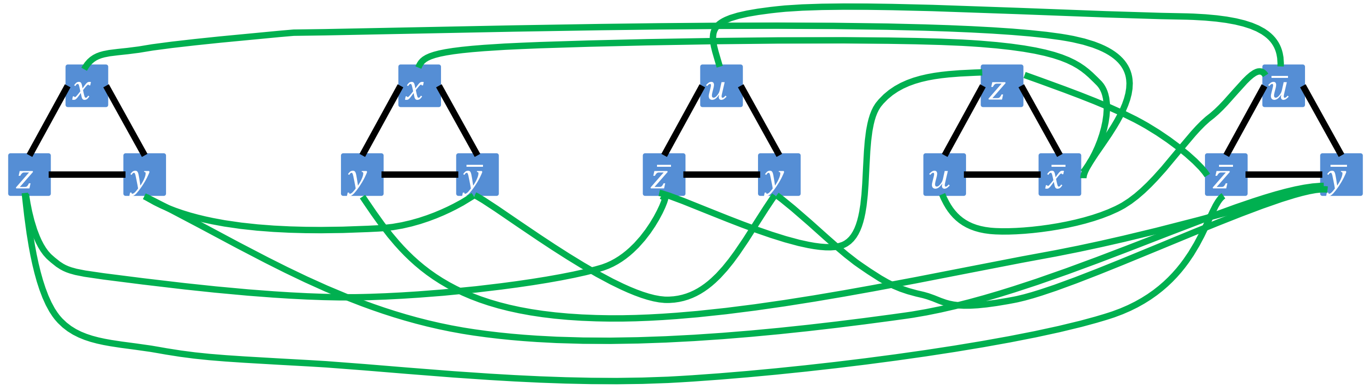




polynomial-time reduction

3-SAT $\leq_p$ $\boldsymbol{k}$-Independent Set

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



For each clause, construct a triangle graph with its three variables as nodes

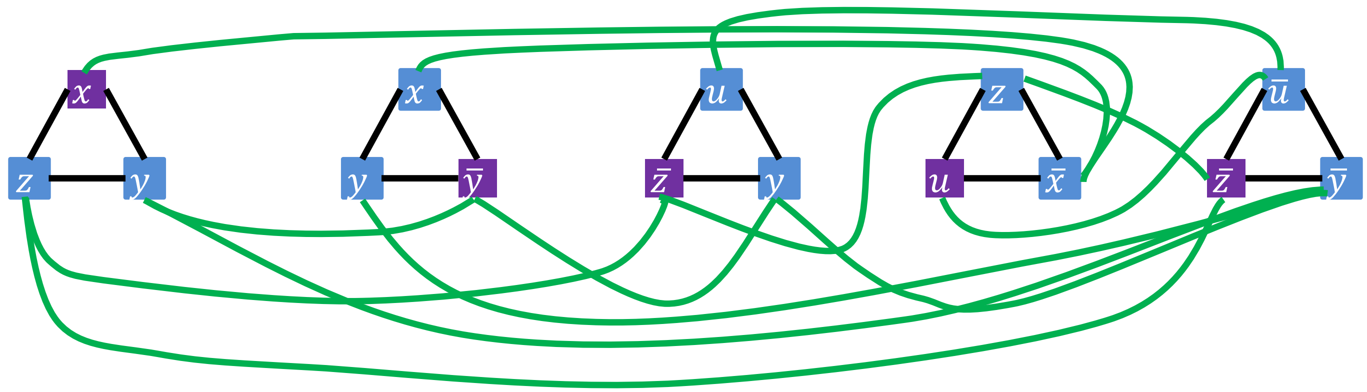
Add an edge between each node and its negation

Let k = number of clauses

Claim. There is a k-independent set in this graph if and only if there is a satisfying assignment

3-SAT $\leq_p$ $\boldsymbol{k}$-Independent Set

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



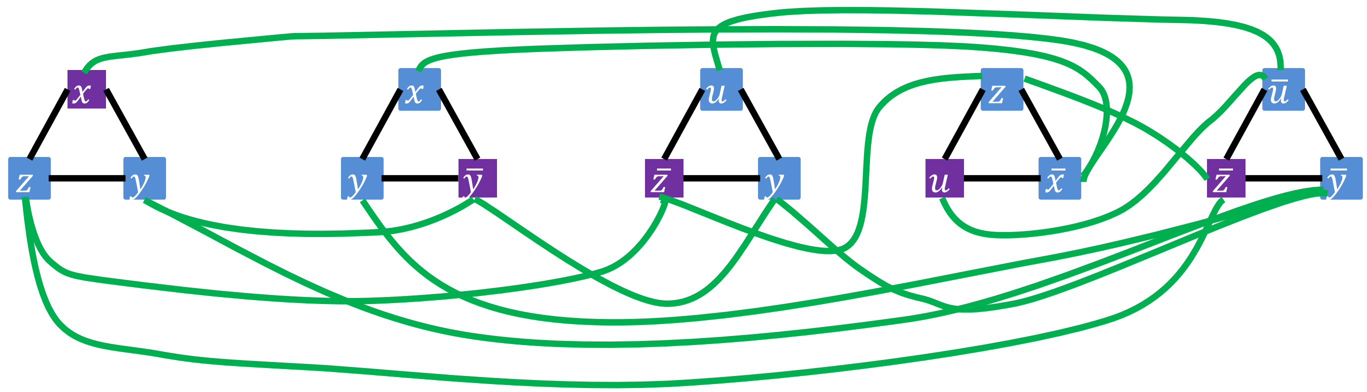
$x = \text{true}$
$y = \text{false}$
$z = \text{false}$
$u = \text{true}$

Suppose there is a k-independent set S in this graph G

- By construction of G, at most one node from each triangle is in S
- Since $|S| = k$ and there are k triangles, each triangle contributes one node
- If a variable x is selected in one triangle, then $\bar{x}$ is never selected in another triangle (since each variable is connected to its negation)
- There are no contradicting assignments, so can set variable chosen in each triangle to “true”; satisfying assignment by construction

3-SAT $\leq_p$ $\boldsymbol{k}$-Independent Set

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$



$x = \text{true}$
$y = \text{false}$
$z = \text{false}$
$u = \text{true}$

Suppose there is a satisfying assignment to the formula

- At least one variable in each clause must be true
- Add the node to that variable to the set S
- There are k clauses, so set S has exactly k nodes
- If we use x in any clause, we will never use $\bar{x}$, so there are no edges among the nodes in S

3-SAT $\leq_p$ $\boldsymbol{k}$-Independent Set

3-SAT

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z})$

x = true
y = false
z = false
u = true

Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

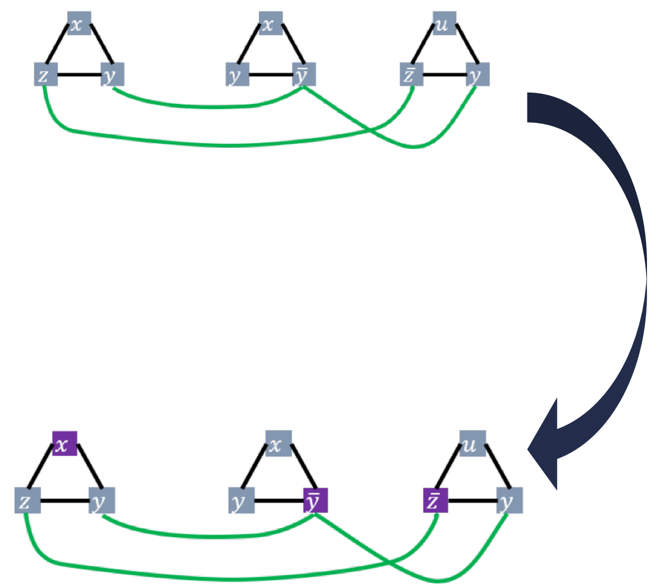
polynomial time

Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

polynomial time

polynomial-time reduction

k-independent set



k-Independent Set is NP-Complete

1. Show that it belongs to NP ✓
2. Show it is NP-Hard ✓
   – Show 3-SAT $\leq_p$ k-independent set

- Next example: *k*-Vertex Cover

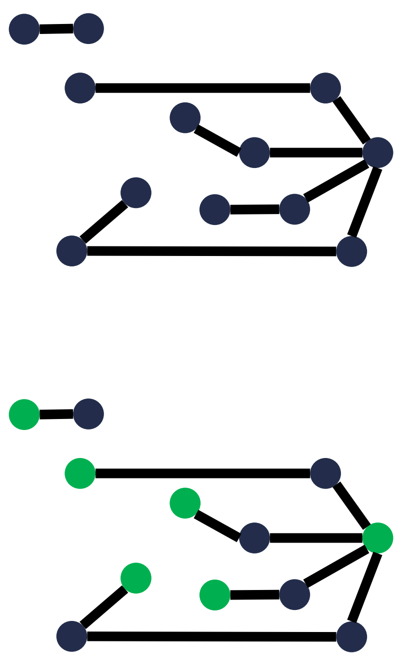
- Remember?
  - We did the following reduction in an earlier slide set!

    k-Independent Set $\leq_p$ *k*-Vertex Cover

  - We just showed *k*-Independent Set is NP-C
  - Therefore…. (you know, right?)

Max Independent Set $\leq_p$ k-Vertex Cover

k-independent set



Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

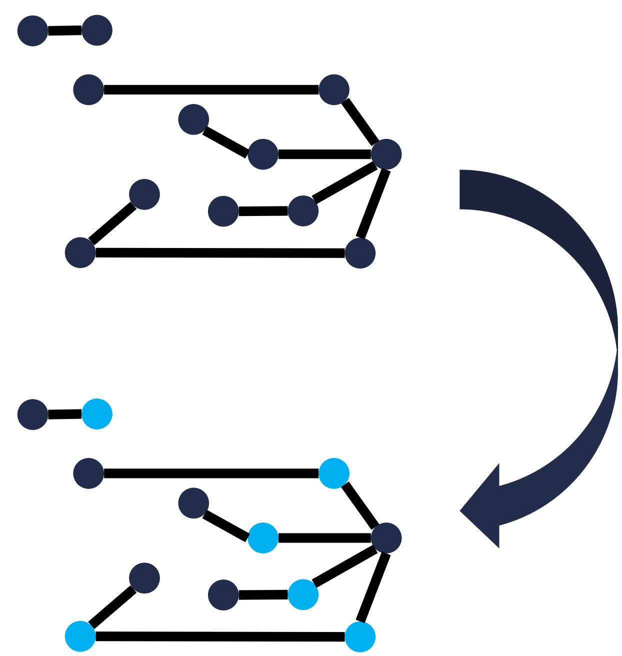
$O(1)$ time

Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

$O(|V|)$ time

Reduction

k-vertex cover



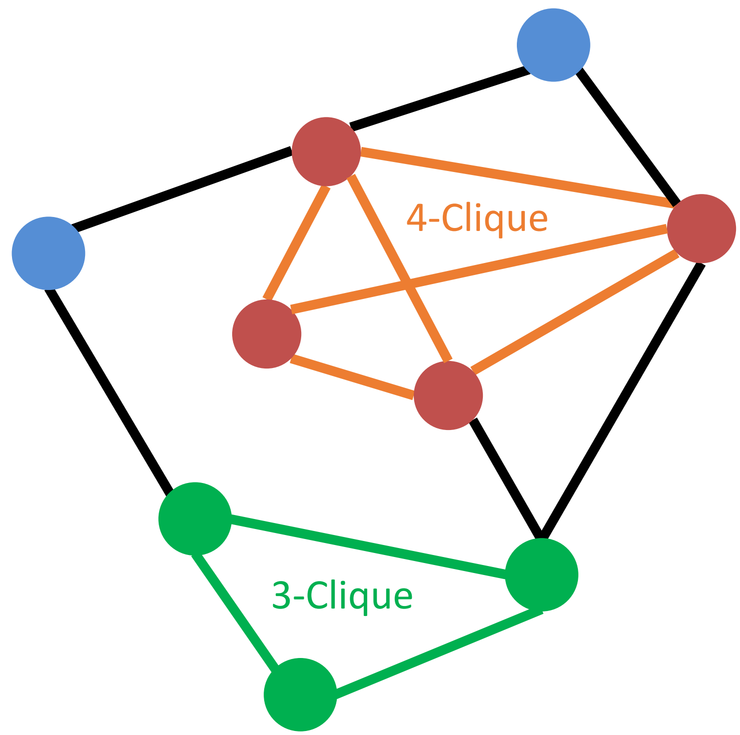
k-Vertex Cover is NP-Complete

1. Show that it belongs to NP
   - Given a candidate cover, check that every edge is covered
2. Show it is NP-Hard
   - Show k-independent set $\leq_p$ k-vertex cover

- Next example: *k*-Clique

k-Clique Problem

- **Clique:** A complete subgraph
- ***k*-Clique problem:** given a graph G and a number k, is there a clique of size k?

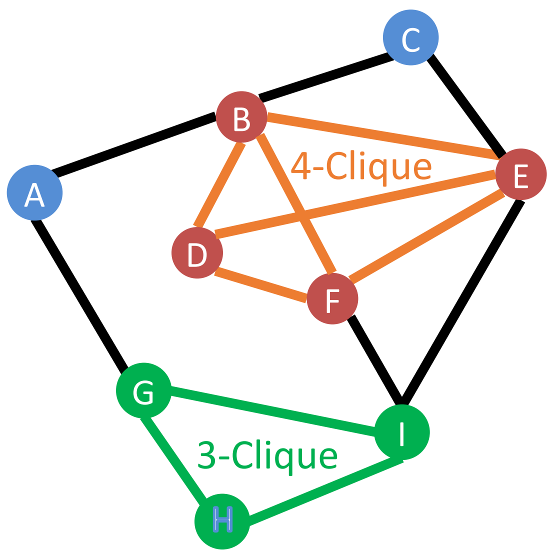


k-Clique is NP-Complete

1. Show that it belongs to NP
   – Give a polynomial time verifier
2. Show it is NP-Hard
   – Give a reduction from a known NP-Hard problem
   – We will show 3-SAT $\leq_p$ k-clique

k-Clique is in NP

- **Show:** For any graph G:
  - There is a short witness ("solution") that G has a k-clique
  - The witness can be checked efficiently (in polynomial time)



Graph G

Suppose $k = 4$

Witness for G: $S = \{B, D, E, F\}$
(nodes in the k-clique)

Checking the witness:

- Check that $|S| = k$ $\quad O(k) = O(|V|)$
- Check that every pair of nodes in S share an edge $\quad O(k^2) = O(|V|^2)$

Total time: $O(|V|^2) = \text{poly}(|V| + |E|)$

k-Clique is NP-Complete

1. Show that it belongs to NP ✅
   – Give a polynomial time verifier
2. Show it is NP-Hard
   – Give a reduction from a known NP-Hard problem
   – We will show 3-SAT $\leq_p$ k-clique

3-SAT $\leq_p$ $\boldsymbol{k}$-Clique

3-SAT

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z})$

x = true
y = false
z = false
u = true

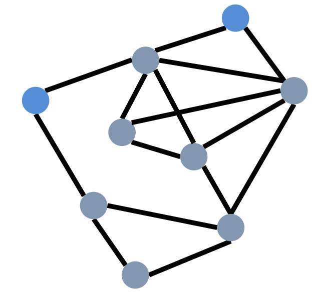
Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

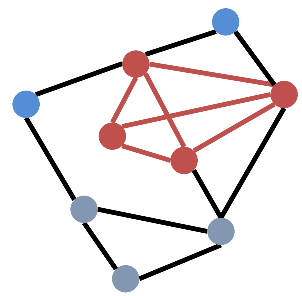
polynomial time

Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

polynomial time

k-clique

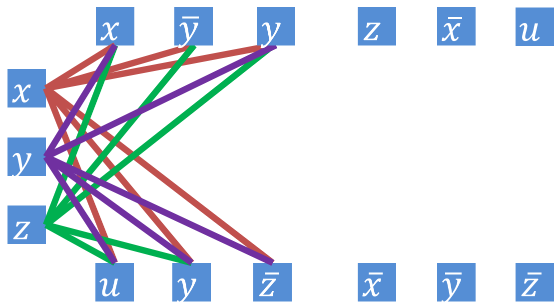




polynomial-time reduction

3-SAT $\leq_p$ $\boldsymbol{k}$-Clique

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



(also do this for the other clauses, omitted due to clutter)

For each clause, introduce a node for each of its three variables

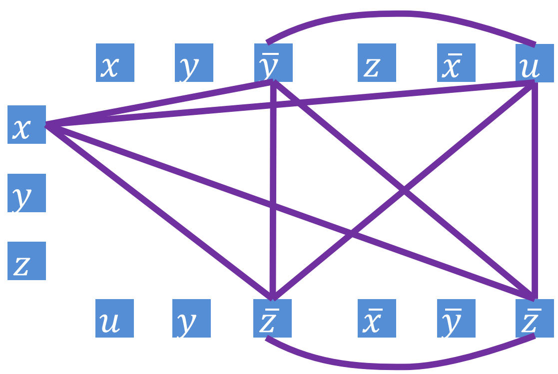
Add an edge from each node to all non-contradictory nodes in the other clauses (i.e., to all nodes that is not the negation of its own variable)

Let $k =$ number of clauses

Claim. There is a k-clique in this graph if and only if there is a satisfying assignment

3-SAT $\leq_p$ $\boldsymbol{k}$-Clique

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$

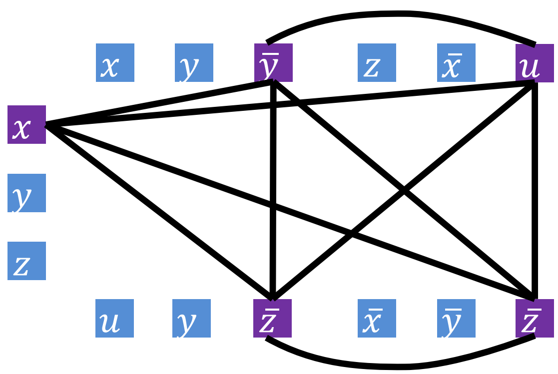


Suppose there is a k-clique in this graph

- There are no edges between nodes for variables in the same clause, so k-clique must contain one node from each clause
- Nodes in clique cannot contain variable and its negation
- Nodes in clique must then correspond to a satisfying assignment

3-SAT $\leq_p$ $\boldsymbol{k}$-Clique

$$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z}) \wedge (z \vee \bar{x} \vee u) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$$



Suppose there is a satisfying assignment to the formula

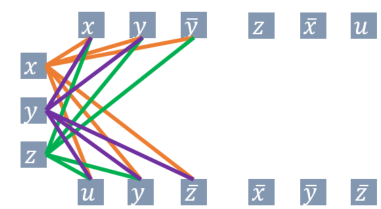
- For each clause, choose one node whose value is true
- There are k clauses, so this yields a collection of k nodes
- Since the assignment is consistent, there is an edge between every pair of nodes, so this constitutes a k-clique

3-SAT $\leq_p$ $\boldsymbol{k}$-Clique

3-SAT

$(x \vee y \vee z) \wedge (x \vee \bar{y} \vee y) \wedge (u \vee y \vee \bar{z})$

k-clique



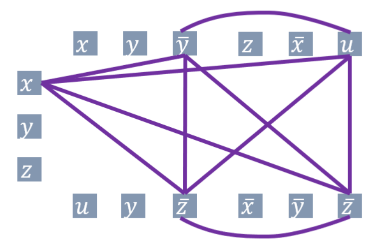
Map instances of problem $\boldsymbol{A}$ to instances of $\boldsymbol{B}$

polynomial time

Map solutions of problem $\boldsymbol{B}$ to solutions of $\boldsymbol{A}$

polynomial time

x = true
y = false
z = false
u = true



polynomial-time reduction

k-Clique is NP-Complete

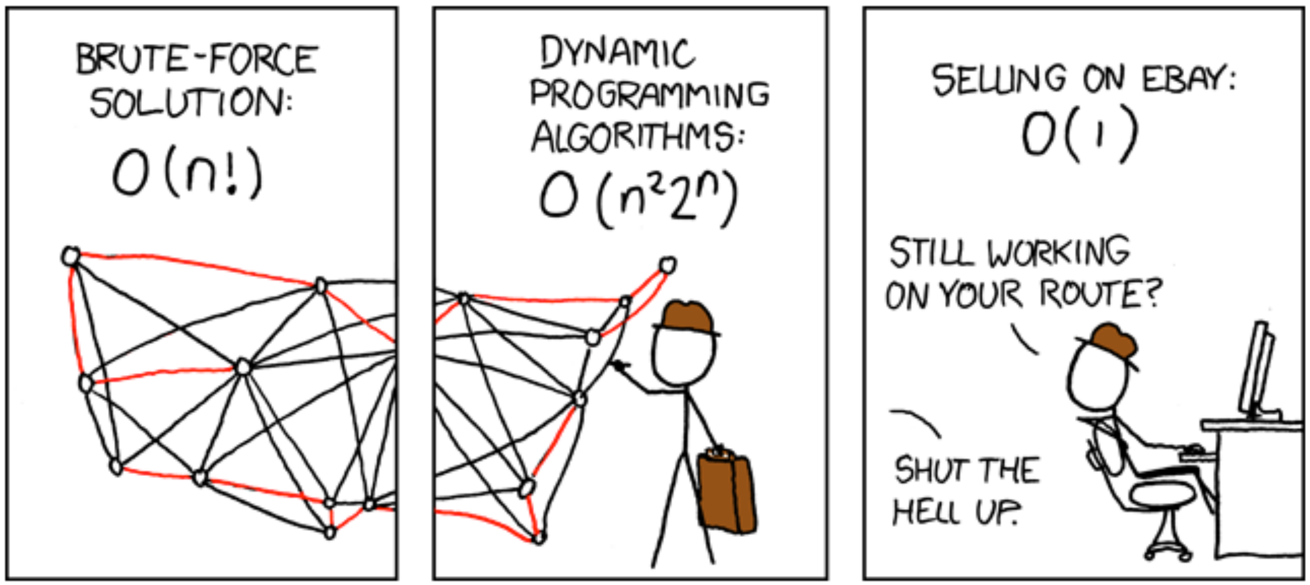
1. Show that it belongs to NP ✅
   – Give a polynomial time verifier
2. Show it is NP-Hard ✅
   – Give a reduction from a known NP-Hard problem
   – We will show 3-SAT $\leq_p$ k-clique

Wrap Up and Reminders

Why Prove NP-Completeness?

- Though nobody has proven that **P ≠ NP**, if you prove a problem NP-Complete, most people accept that it is probably exponential
- Therefore it can be important for you to prove that a problem is NP-Complete
  - Don't need to try to come up perfect non-exponential algorithm
  - Can instead work on *approximation algorithms*

What's a poor salesperson to do?



http://xkcd.com/399/

Approximation Algorithms

- Look at first 3 pages of Ch. 35 of CLRS textbook
- Can we find an algorithm for problem $A \in$ **NP-C** that:
  - Runs in polynomial time
  - Gets "near optimal" results
- Prove some bound on the algorithm's correctness in terms of the true optimal result
  - No worse that (some factor) of optimal
  - "It's not always right (best), but it's guaranteed to be this close."

General Comments

- At least 3000 problems have been shown to be NP-Complete
  - That number is from a non-recent report, so we might say that counts is a weak lower-bound on the true number found
  - https://en.wikipedia.org/wiki/List_of_NP-complete_problems including some popular games
- Some reductions are profound, some are comparatively easy, many are easy once the key insight is given

Other NP-Complete Problems

- *Hamilton Path/Cycle, Traveling Salesperson*
- *Subset-sum*: Given a set of integers, does there exist a subset that adds up to some target T ?
- *0-1 knapsack*: when weights not just integers
- *Graph coloring*: can a given graph be colored with k colors such that no adjacent vertices are the same color?
- Etc…

Review (Again)

- A problem B is *NP-complete*
  - if it is in NP **and** it is NP-hard.
- A problem B is *NP-hard*
  - if *every* problem in NP is reducible to **B**.
- A problem A is *reducible* to a problem B if
  - there exists a polynomial reduction function T such that
    - For every string x,
    - if x is a yes input for A, then T(x) is a yes input for B
    - if x is a no input for A, then T(x) is a no input for B.
    - T can be computed in polynomially bounded time.

"Consequences" of NP-Completeness

- NP-Complete the set of the"hardest" problems in NP, with these important properties:
  - If any *one* NP-Complete problem can be solved in polynomial time…
  - …then *every* NP-Complete problem can be solved in polynomial time…
  - …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
  - Or, prove an exponential lower-bound for *any single* **NP-C** problem, then *every* **NP-C** problem is exponential

Therefore: solve (say) traveling salesperson problem in O(n^{100}) time, you've proved that **P = NP**. Retire rich & famous!

What We Don't Know: Open Questions

- Is it **impossible** to solve an NP-c problem in polynomial time?
  - No one has proved an exponential lower bound for any problem in NP.
  - But, most computer scientists believe such a lower bound exists for NP-c problems.
- Are all problems in NP tractable or intractable?
  I.e., does P=NP or not?
  - If someone found a polynomial solution to any NP-c problem, we'd know P = NP.
  - But, most computer scientists believe P≠ NP.