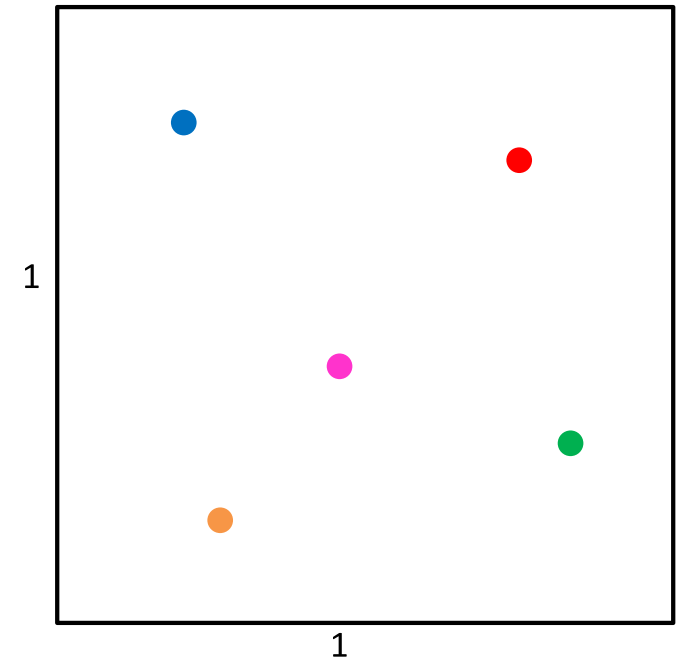
CS4102 Algorithms

Spring 2020

Warm up

Given any 5 points on the unit square, show there's always a pair distance $\leq \frac{\sqrt{2}}{2}$ apart

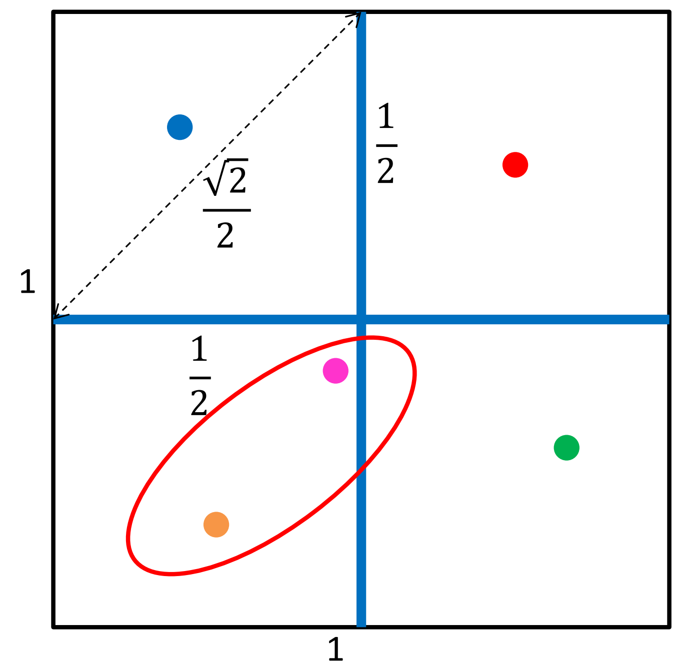


CS4102 Algorithms
Spring 2020

If points p_1, p_2 in same quadrant, then $\delta(p_1, p_2) \leq \frac{\sqrt{2}}{2}$

Given 5 points, two must share the same quadrant

Pigeonhole Principle!



Today's Keywords

- Karatsuba (one last time!)
- Solving recurrences
- Cookbook Method
- Master Theorem
- Substitution Method

CLRS Readings

- Chapter 4

Homeworks

- Hw1 due tomorrow at 11pm
  - Written (use Latex!) – Submit BOTH pdf and zip!
  - Asymptotic notation
  - Recurrences
  - Divide and Conquer

Recurrence Solving Techniques

Tree

? ✓ Guess/Check (induction)

“Cookbook”

Substitution

Induction (review)

Goal: $\forall k \in \mathbb{N}, P(k)$ holds

Base case(s): $P(1)$ holds

Technically, called *strong induction*

Hypothesis: $\forall x \le x_0, P(x)$ holds

Inductive step: show $P(1), \ldots, P(x_0) \Rightarrow P(x_0 + 1)$

Guess and Check Intuition

- **Show:** $T(n) \in O(g(n))$
- **Consider:** $g_*(n) = c \cdot g(n)$ for some constant c, i.e. pick $g_*(n) \in O(g(n))$
- **Goal:** show $\exists n_0$ such that $\forall n > n_0, T(n) \le g_*(n)$
  - (definition of big-O)
- **Technique:** Induction
  - Base cases:
    - show $T(1) \le g_*(1), T(2) \le g_*(2)$, … for a small number of cases (may need additional base cases)
  - Hypothesis:
    - $\forall n \le x_0, T(n) \le g_*(n)$
  - Inductive step:
    - Show $T(x_0 + 1) \le g_*(x_0 + 1)$

Need to ensure that in inductive step, can either appeal to a base case or to the inductive hypothesis

Karatsuba Algorithm

$$\begin{array}{r} a\ \ b \\ \times\ c\ \ d \\ \hline \end{array}$$

1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^{n}(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

<u>Pseudo-code</u>

1. $x \leftarrow$ Karatsuba(a, c)
2. $y \leftarrow$ Karatsuba(b, d)
3. $z \leftarrow$ Karatsuba$(a+b, c+d) - x - y$
4. Return $10^{n}x + 10^{n/2}z + y$

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

$$T(n) = 24\left(n^{\log_2 3}\right) - 16n \;= \Theta(n^{\log_2 3})$$

$$\approx \Theta(n^{1.585})$$

Karatsuba Guess and Check

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal: $T(n) \le 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base cases: by inspection, holds for small n (at home)

Hypothesis: $\forall n \le x_0, T(n) \le 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \le 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

Karatsuba Guess and Check

Karatsuba Guess and Check

What if we leave out the $-16n$?

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal: $T(n) \leq 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base cases: by inspection, holds for small n (at home)

Hypothesis: $\forall n \leq x_0, T(n) \leq 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

What we wanted: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3}$ Induction failed!

What we got: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} + 8(x_0 + 1)$

"Bad Mergesort" Guess and Check

$$T(n) = 2\,T\left(\frac{n}{2}\right) + 209n$$

Goal: $T(n) \leq 209n \log_2 n \; = O(n \log_2 n)$

Base cases: $T(1) = 0$

$T(2) = 518 \leq 209 \cdot 2 \log_2 2$

… up to some small k

Hypothesis: $\forall n \leq x_0, T(n) \leq 209n \log_2 n$

Inductive step: $T(x_0 + 1) \leq 209(x_0 + 1) \log_2(x_0 + 1)$

Prove this on your own

Recurrence Solving Techniques

Tree

Guess/Check

"Cookbook"

Substitution

Observation

- **Divide**: $D(n)$ time
- **Conquer**: recurse on small problems, size s
- **Combine**: $C(n)$ time
- **Recurrence:**

$$T(n) = D(n) + \sum T(s) + C(n)$$

- Many D&C recurrences are of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n), \qquad \text{where } f(n) = D(n) + C(n)$$

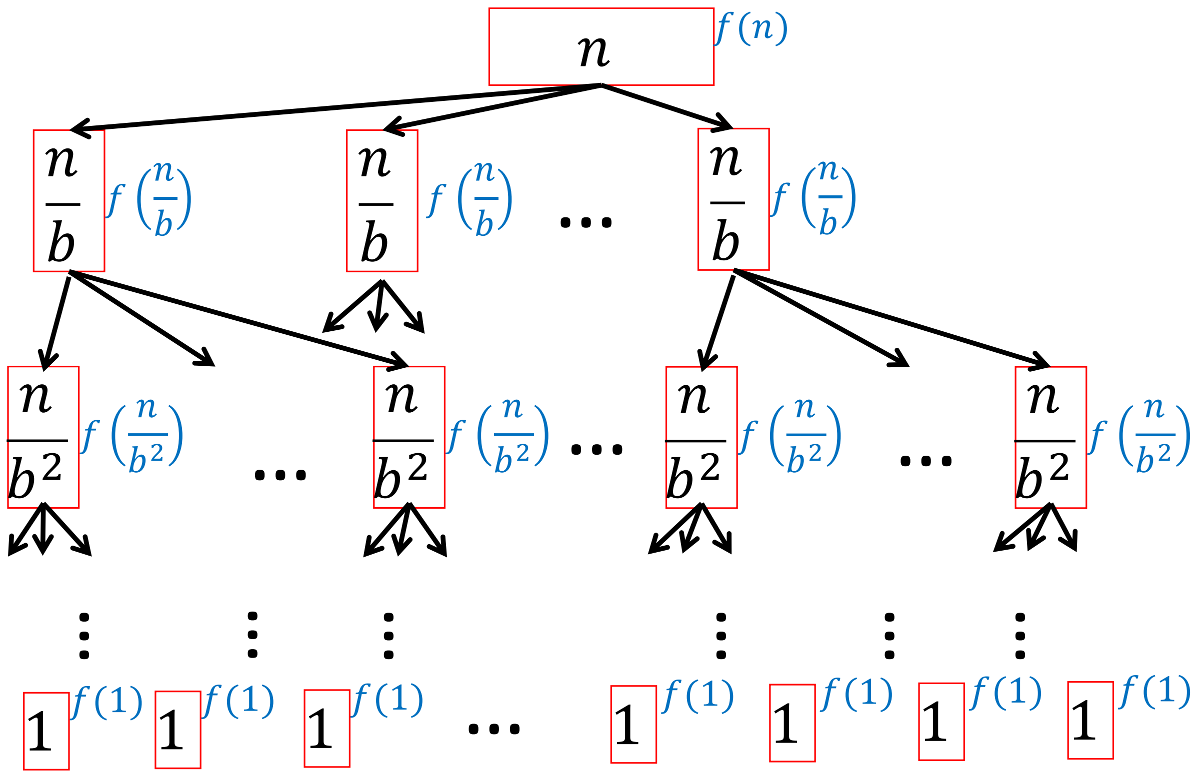
Remember…

- Better Attendance: $T(n) = T\left(\frac{n}{2}\right) + 2$
- MergeSort: $T(n) = 2\,T\left(\frac{n}{2}\right) + n$
- D&C Multiplication: $T(n) = 4T\left(\frac{n}{2}\right) + 5n$
- Karatsuba: $T(n) = 3T\left(\frac{n}{2}\right) + 8n$

General

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)$$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



$$f(n)$$

$$af\left(\frac{n}{b}\right)$$

$$a^2 f\left(\frac{n}{b^2}\right)$$

$$\vdots$$

$$a^{\log_b n} f(1)$$

3 Cases

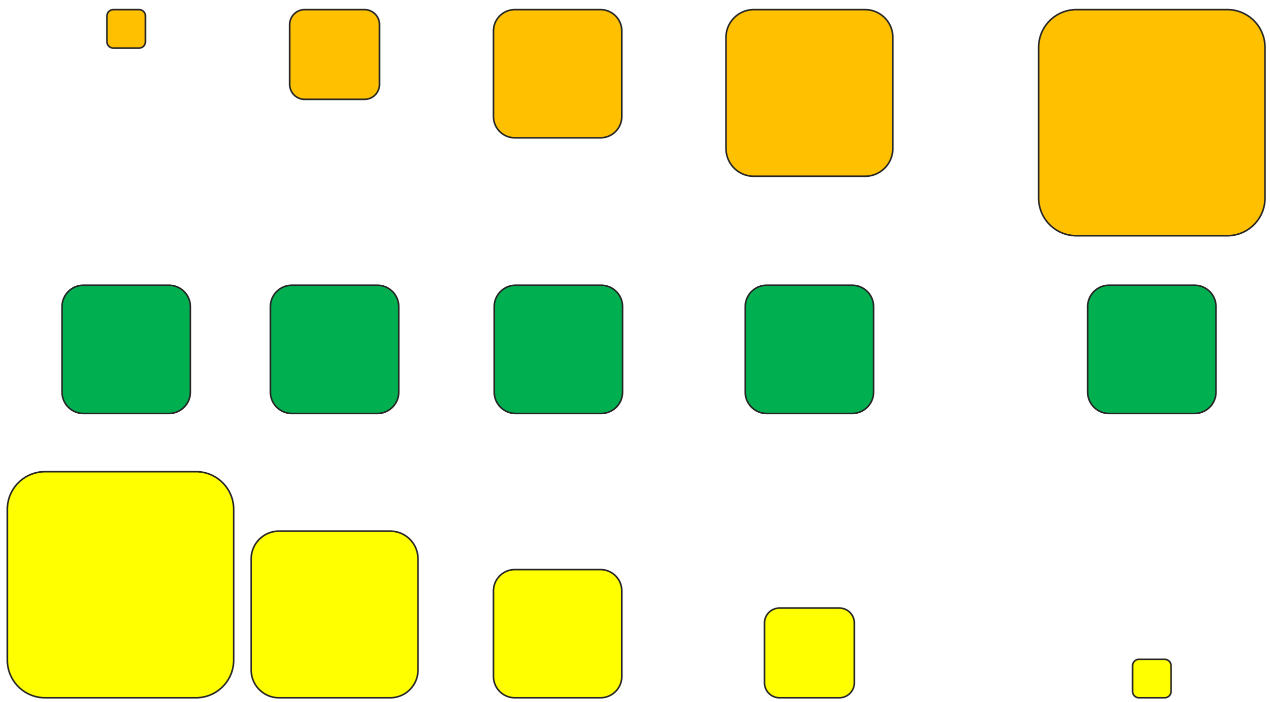
$$L = \log_b n$$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

Case 1:
Most work happens at the leaves

Case 2:
Work happens consistently throughout

Case 3:
Most work happens at top of tree



Master Theorem

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$,

then $T(n) = \Theta\left(n^{\log_b a}\right)$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$,

and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$

and all sufficiently large n,

then $T(n) = \Theta(f(n))$

Proof of Case 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right),$$

$$f(n) = O\left(n^{\log_b a - \varepsilon}\right) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

Insert math here…

Proof of Case 1

Proof of Case 1

Conclusion: $T(n) = O(n^{\log_b a})$

Master Theorem Example 1

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

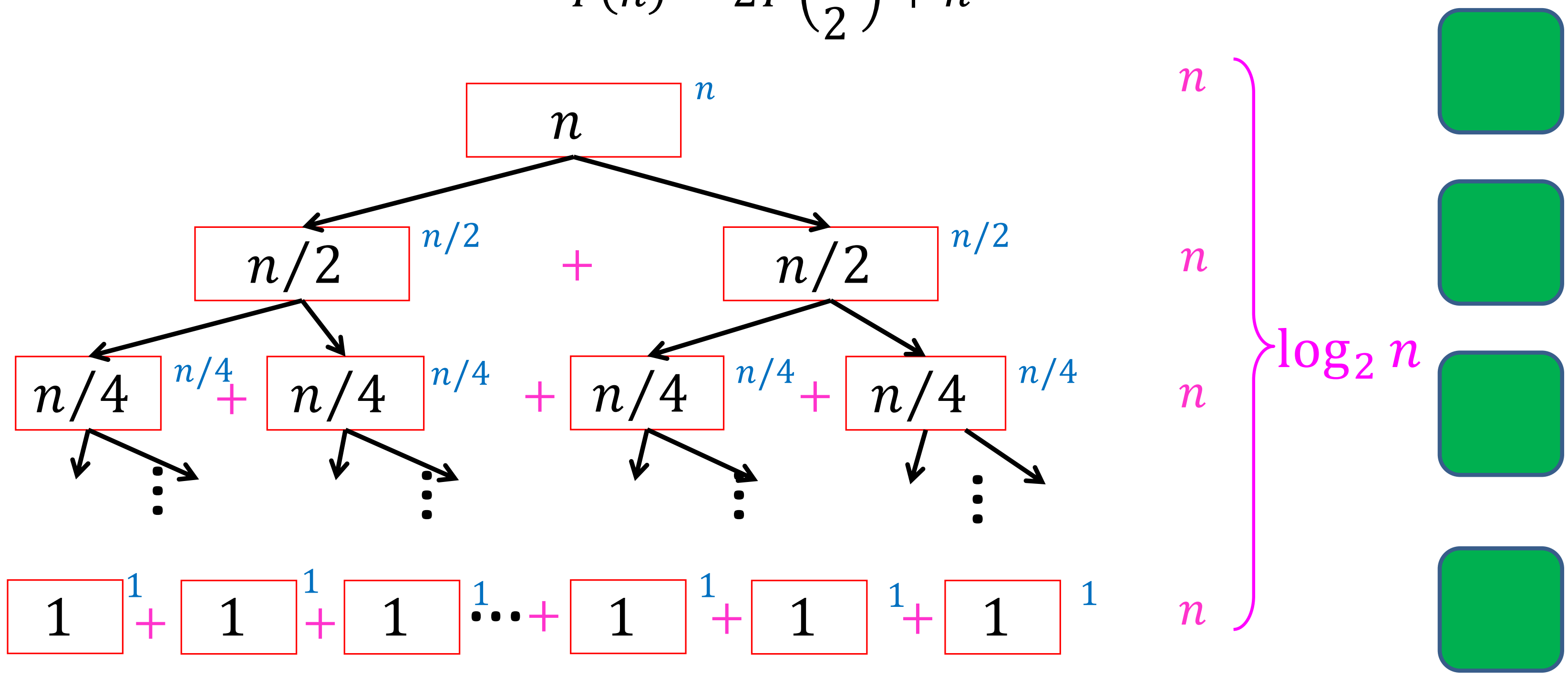
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Case 2

$$\Theta\left(n^{\log_2 2} \log n\right) = \Theta(n \log n)$$

Tree method

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



Master Theorem Example 2

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

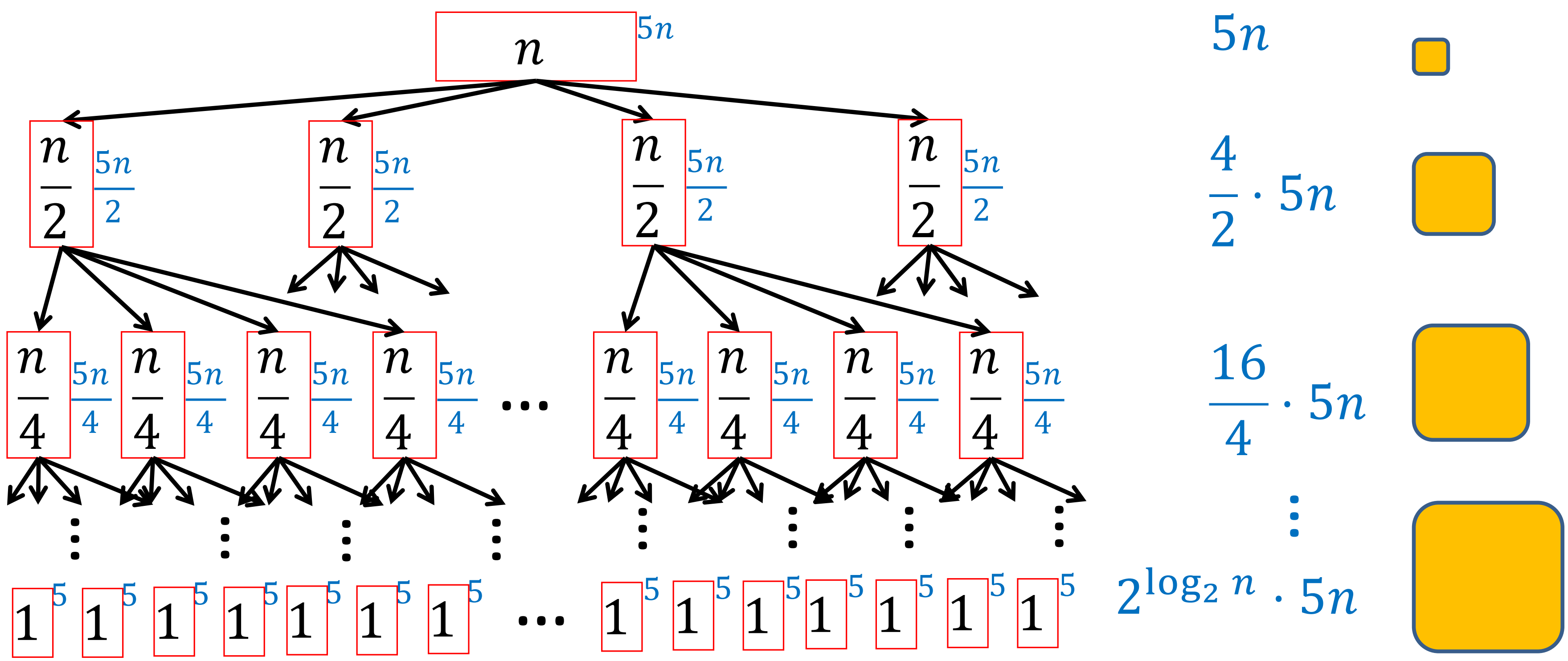
$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Case 1

$$\Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$



Tree method

$$T(n) = 4T\left(\frac{n}{2}\right) + 5n$$

Cost is <u>increasing</u> with the recursion depth (due to large number of subproblems)

Most of the work happening in the leaves

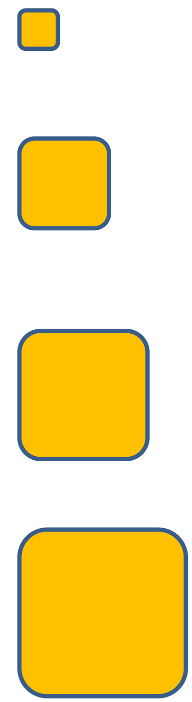
$5n$

$\frac{4}{2} \cdot 5n$

$\frac{16}{4} \cdot 5n$

$\vdots$

$2^{\log_2 n} \cdot 5n$



Master Theorem Example 3

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

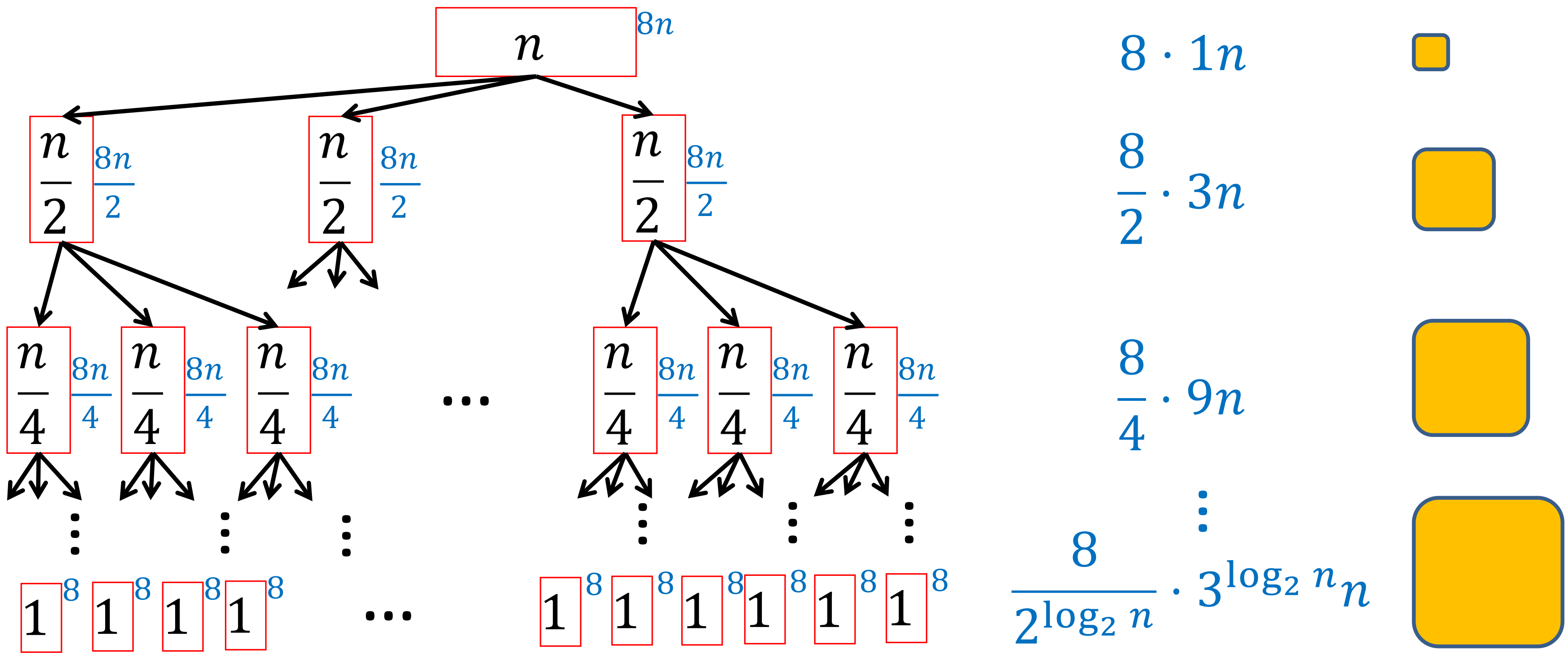
$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Case 1

$$\Theta\left(n^{\log_2 3}\right) \approx \Theta(n^{1.5})$$

Karatsuba

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$



Master Theorem Example 4

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

- Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- Case 2: if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$
- Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$

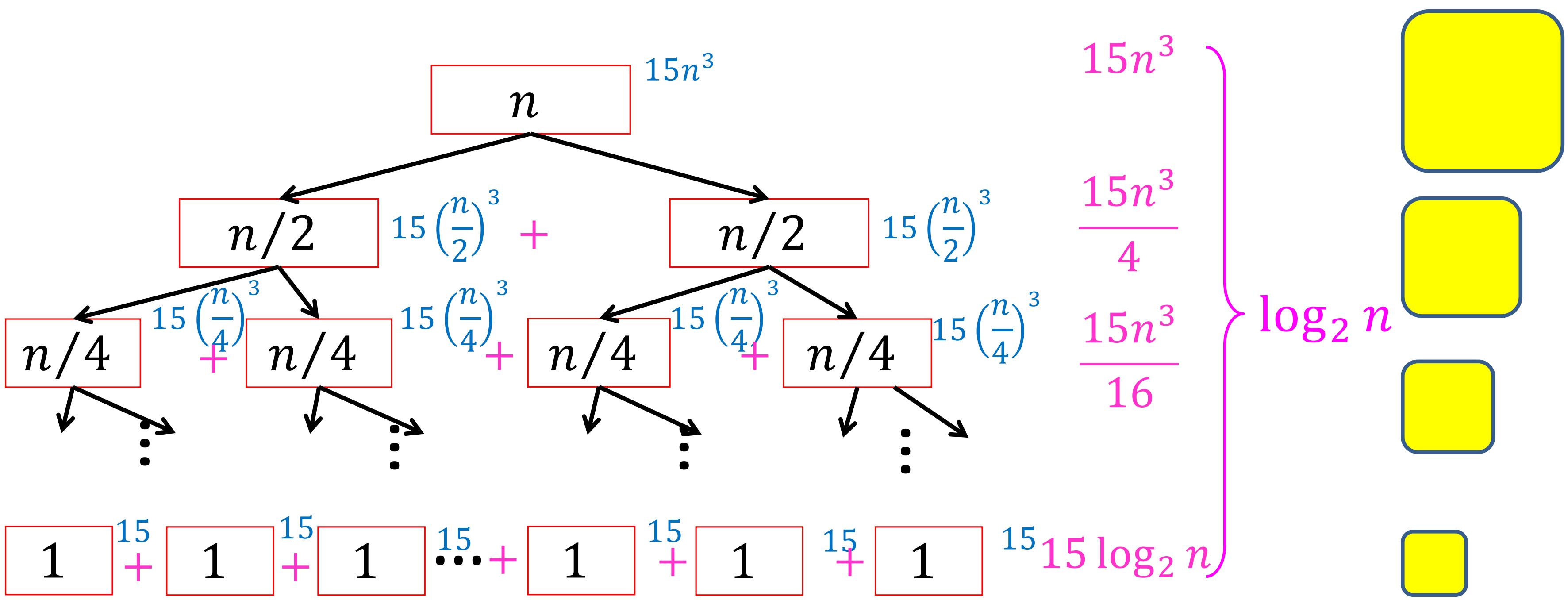
$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

Case 3

$$\Theta(n^3)$$

Tree method

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

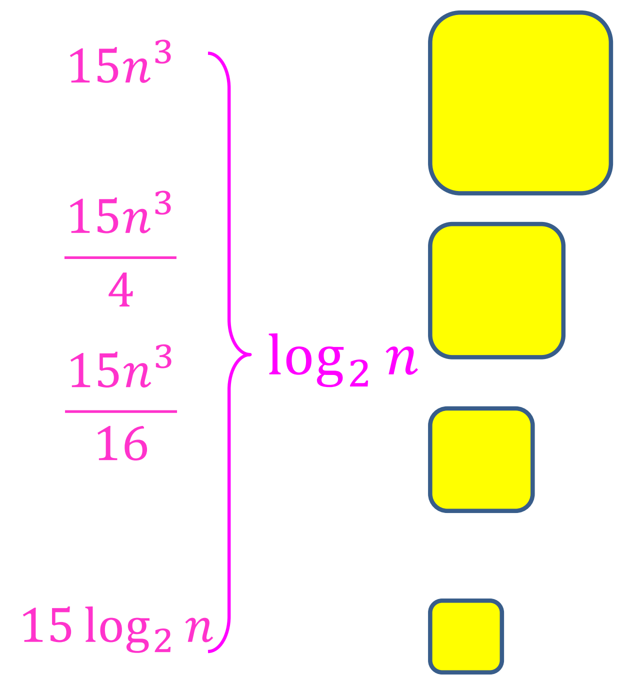


Tree method

$$T(n) = 2T\left(\frac{n}{2}\right) + 15n^3$$

Cost is <u>decreasing</u> with the recursion depth (due to high *non-recursive* cost)

Most of the work happening at the top



Recurrence Solving Techniques

Tree

Guess/Check

“Cookbook”

Substitution

Substitution Method

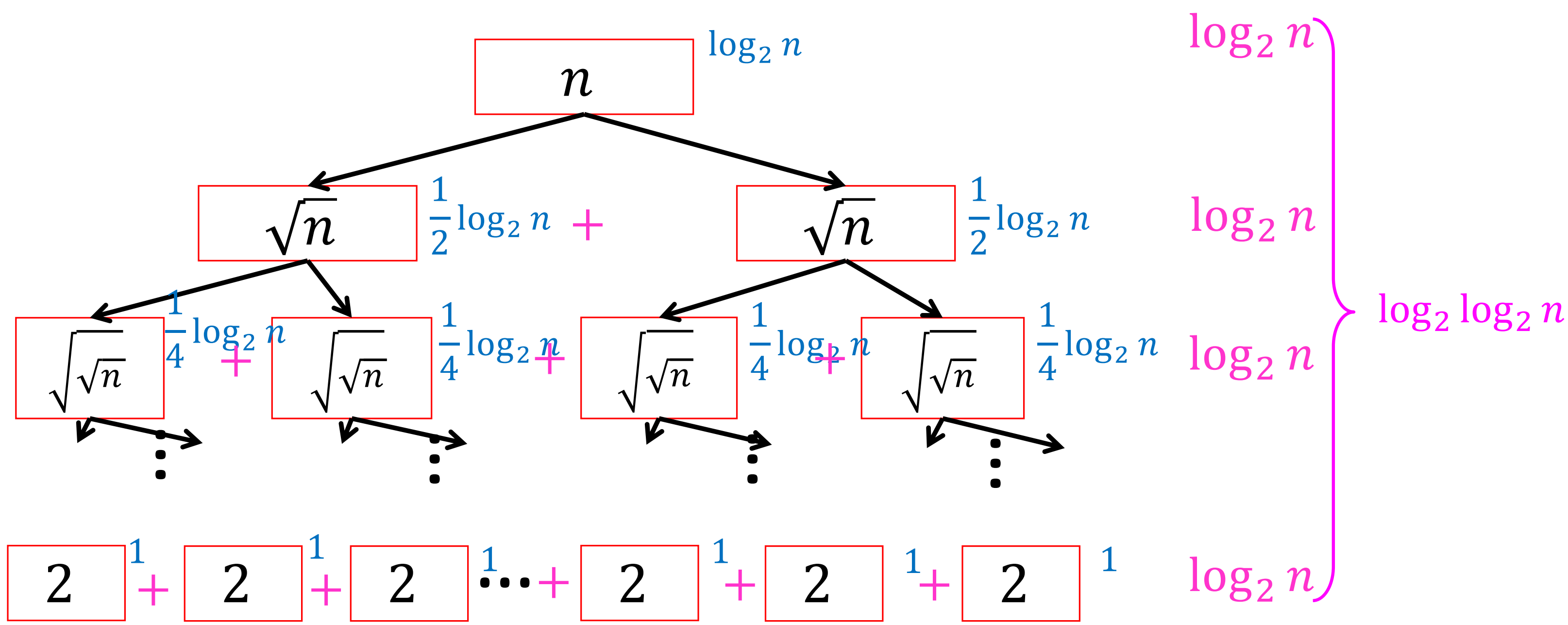
- Idea: take a “difficult” recurrence, re-express it such that one of our other methods applies.
- Example:

$$T(n) = 2T(\sqrt{n}) + \log_2 n$$

Tree method

$$\log_2 n^{1/2} = \frac{1}{2}\log_2 n$$

$$T(n) = 2T(\sqrt{n}) + \log_2 n$$



$$T(n) = O(\log_2 n \cdot \log_2 \log_2 n)$$

Substitution Method

$$T(n) = 2T(\sqrt{n}) + \log_2 n$$
$$= 2T\left(n^{1/2}\right) + \log_2 n$$

I don't like the ½ in the exponent

Let $n = 2^m$, i.e. $m = \log_2 n$

Now the variable is in the exponent on both sides!

$T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m$ Rewrite in terms of exponent!

Let $S(m) = 2S\left(\frac{m}{2}\right) + m$ Case 2!

Let $S(m) = \Theta(m \log m)$ Substitute Back

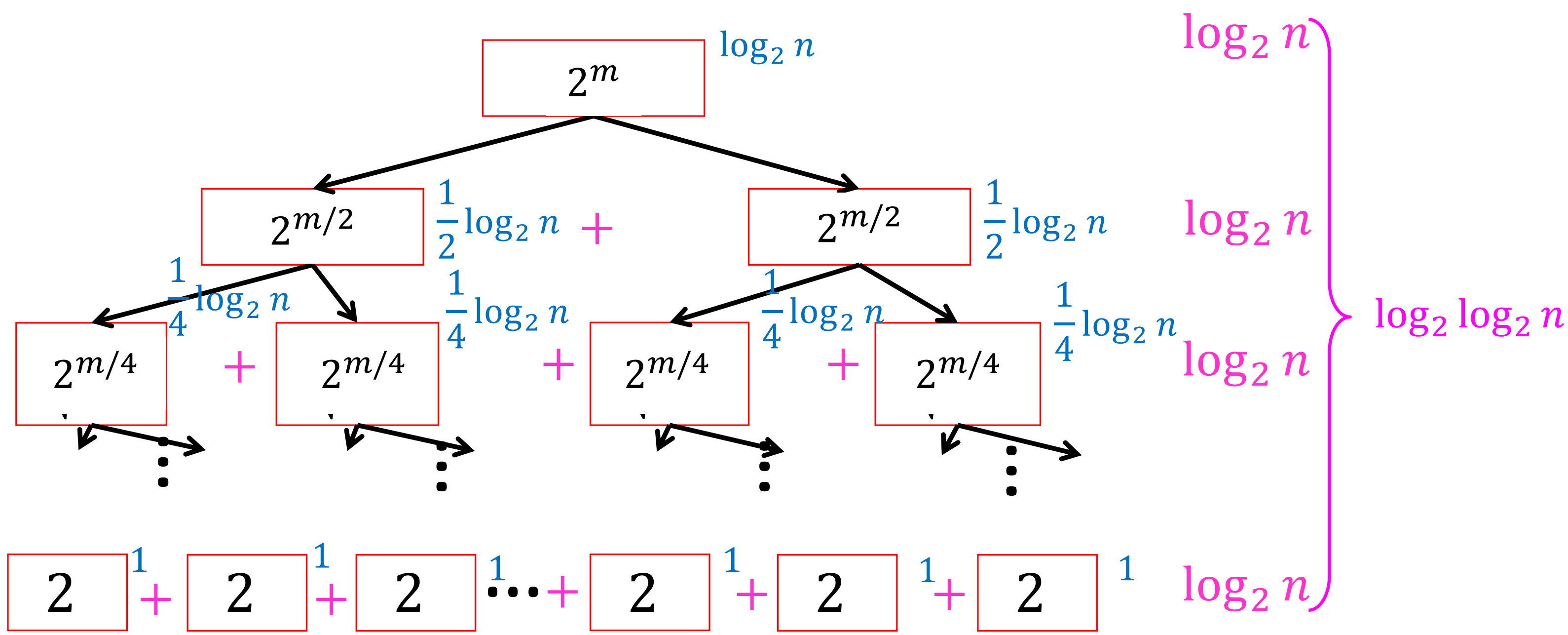
S will operate exactly as T, just redefined in terms of the exponent

$S(m) = T(2^m)$

Let $\mathrm{T}(n) = \Theta(\log n \log \log n)$

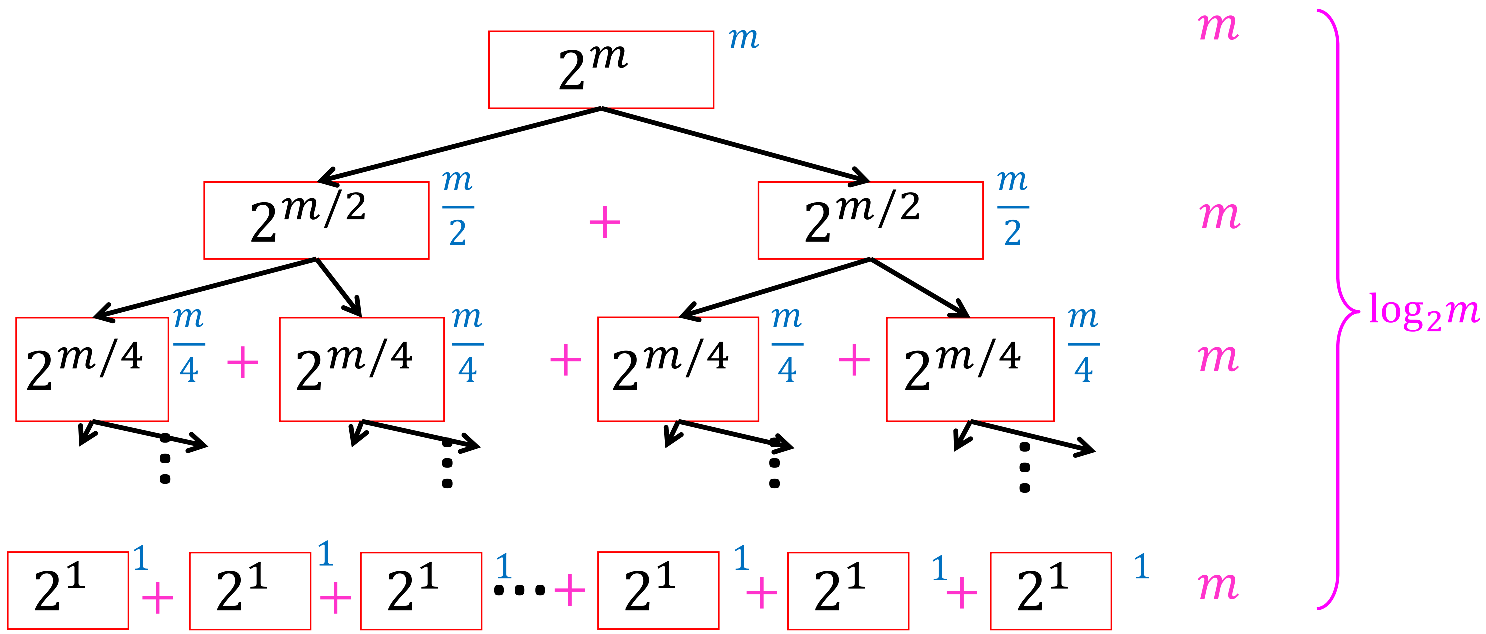
Tree method

$$n = 2^m \qquad T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m$$



Tree method

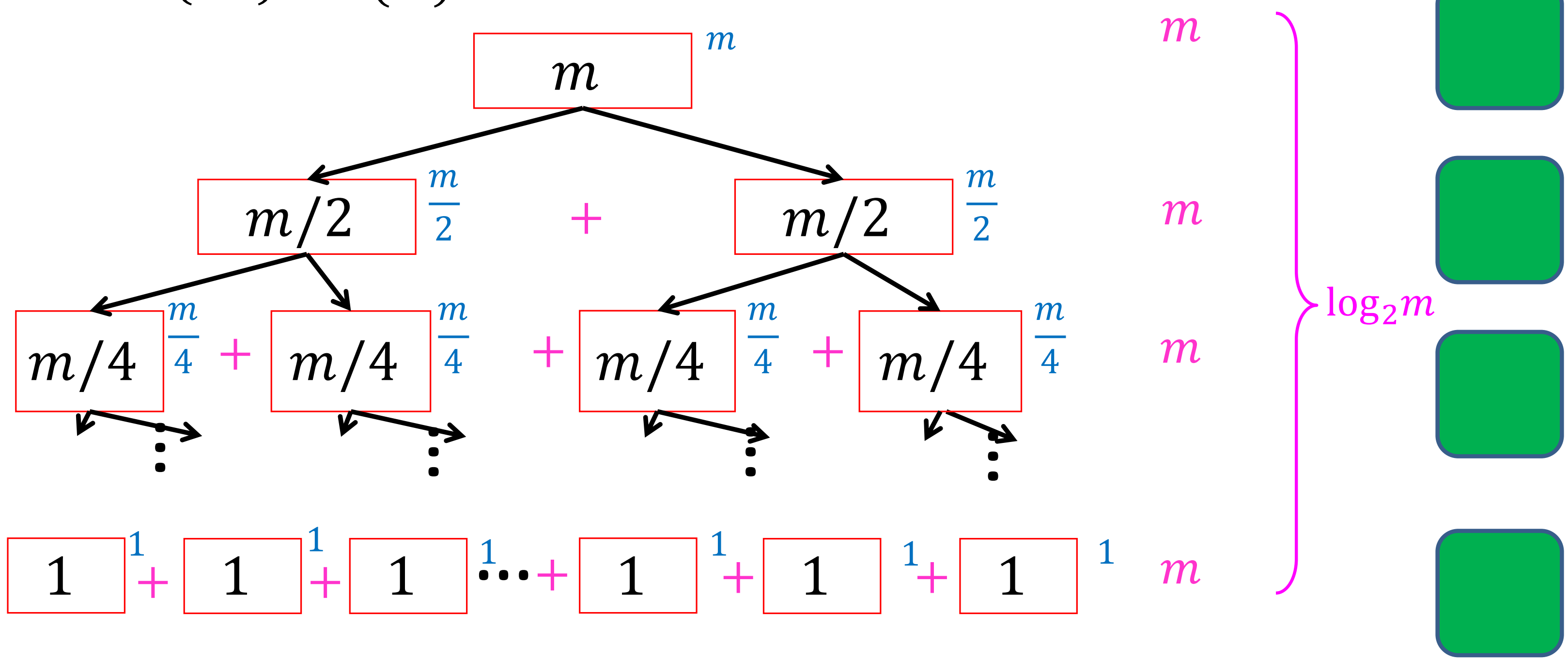
$n = 2^m$ $\qquad T(2^m) = 2T(2^{m/2}) + m$



Tree method

$$n = 2^m$$
$$T(2^m) = S(m)$$

$$S(m) = 2S\left(\frac{m}{2}\right) + m$$



$$T(n) = O(m \cdot \log_2 m) = O(\log_2 n \cdot \log_2 \log_2 n)$$