CS4102 Algorithms

Spring 2020

Warm up

Decode the line below into English

(hint: use Google or Wolfram Alpha)

→ .. .-.. .. -.- . .- .-.. --. --- .-. .. - .... -- ...

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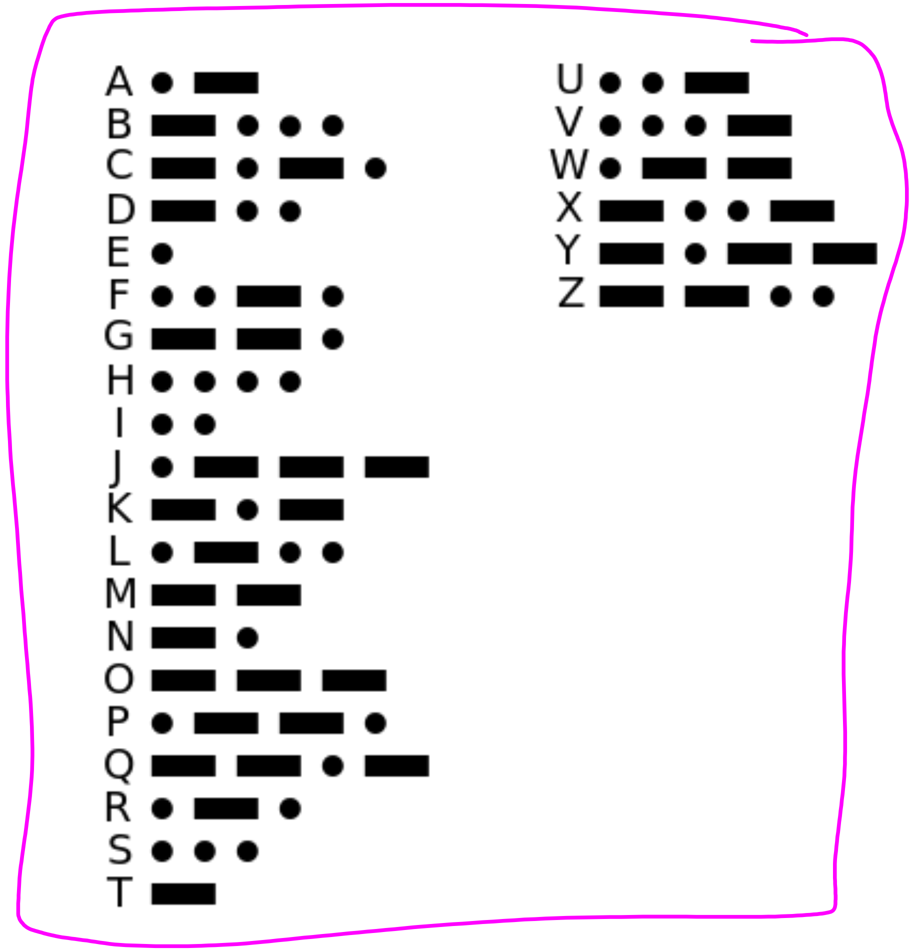


A	• ▬
B	▬ • • •
C	▬ • ▬ •
D	▬ • •
E	•
F	• • ▬ •
G	▬ ▬ •
H	• • • •
I	• •
J	• ▬ ▬ ▬
K	▬ • ▬
L	• ▬ • •
M	▬ ▬
N	▬ •
O	▬ ▬ ▬
P	• ▬ ▬ •
Q	▬ ▬ • ▬
R	• ▬ •
S	• • •
T	▬
U	• • ▬
V	• • • ▬
W	• ▬ ▬
X	▬ • • ▬
Y	▬ • ▬ ▬
Z	▬ ▬ • •

→ •• •−•• •• −•− • •− •−•• −−• −−− •−• •• − •••• −− •••

Today's Keywords

- Greedy Algorithms
- Exchange Argument
- Choice Function
- Prefix-free code
- Compression
- Huffman Code

CLRS Readings

- Chapter 16

Homeworks

- HW6 Due Sunday, April 5 @ 11pm
  - Written (use latex)
  - DP and Greedy
- EC1 - optional homework
  - No office hours for that assignment
- HW4 grades coming later this week

Greedy Algorithms

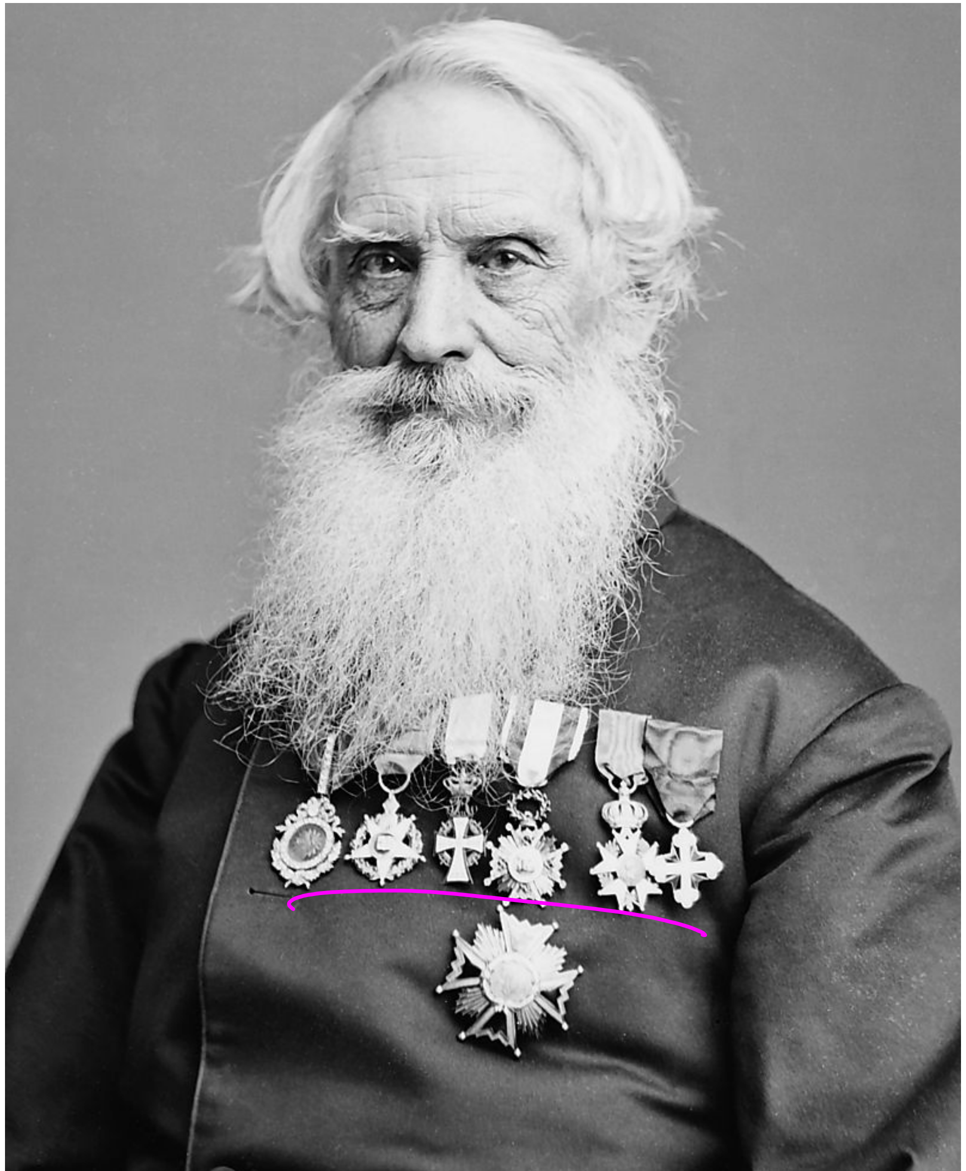
- Require Optimal Substructure
  - – Solution to larger problem contains the solution to a smaller one
  - – Only one subproblem to consider!
- Idea:
  1. Identify a greedy choice property
     - How to make a choice guaranteed to be included in some optimal solution
  2. Repeatedly apply the choice property until no subproblems remain

Exchange argument

- Shows correctness of a greedy algorithm
- Idea:
  - Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
  - How to show my sandwich is at least as good as yours:
    - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

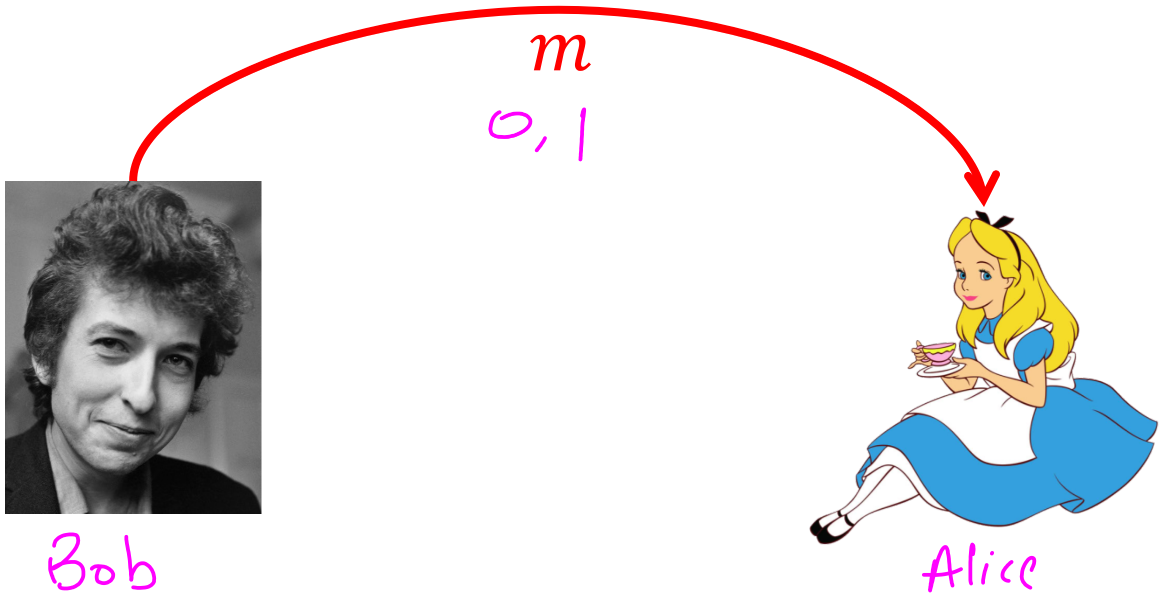
Sam Morse

- Engineer and artist



Message Encoding

- Problem: need to electronically send a message to two people at a distance.
- Channel for message is binary (either on or off)



How can we do it?

wiggle, wiggle, wiggle like a gypsy queen
wiggle, wiggle, wiggle all dressed in green

- Take the message, send it over character-by-character with an encoding

4bits/char

Character Frequency	Encoding
a: 2	0000
d: 2	0001
e: 13	0010
g: 14	0011
i: 8	0100
k: 1	0101
l: 9	0110
n: 3	0111
p: 1	1000
q: 1	1001
r: 2	1010
s: 3	1011
u: 1	1100
w: 6	1101
y: 2	1110

15

How efficient is this?

68

wiggle wiggle wiggle like a gypsy queen
wiggle wiggle wiggle all dressed in green

Each character requires 4 bits

$$\ell_c = 4$$

Cost of encoding:

$$B(T, \{f_c\}) = \sum_{character\ c} \ell_c f_c = 68 \cdot 4 = 272$$

Character Frequency	Encoding
a: 2	0000
d: 2	0001
e: 13	0010
g: 14	0011
i: 8	0100
k: 1	0101
l: 9	0110
n: 3	0111
p: 1	1000
q: 1	1001
r: 2	1010
s: 3	1011
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y: 2	1110

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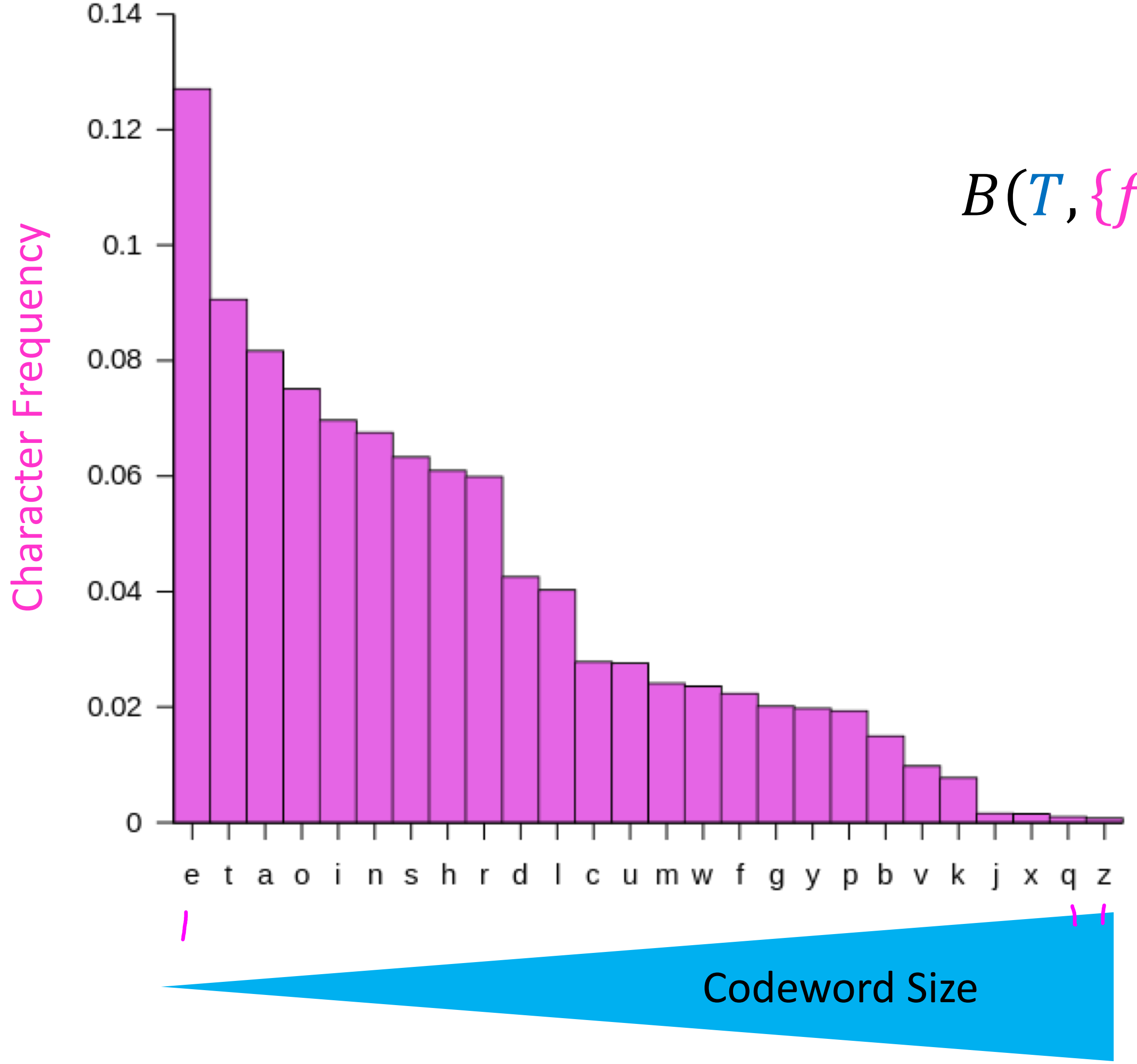
q: 1	1001
r: 2	1010
s: 3	1011
u: 1	1100
w: 6	1101
y: 2	1110

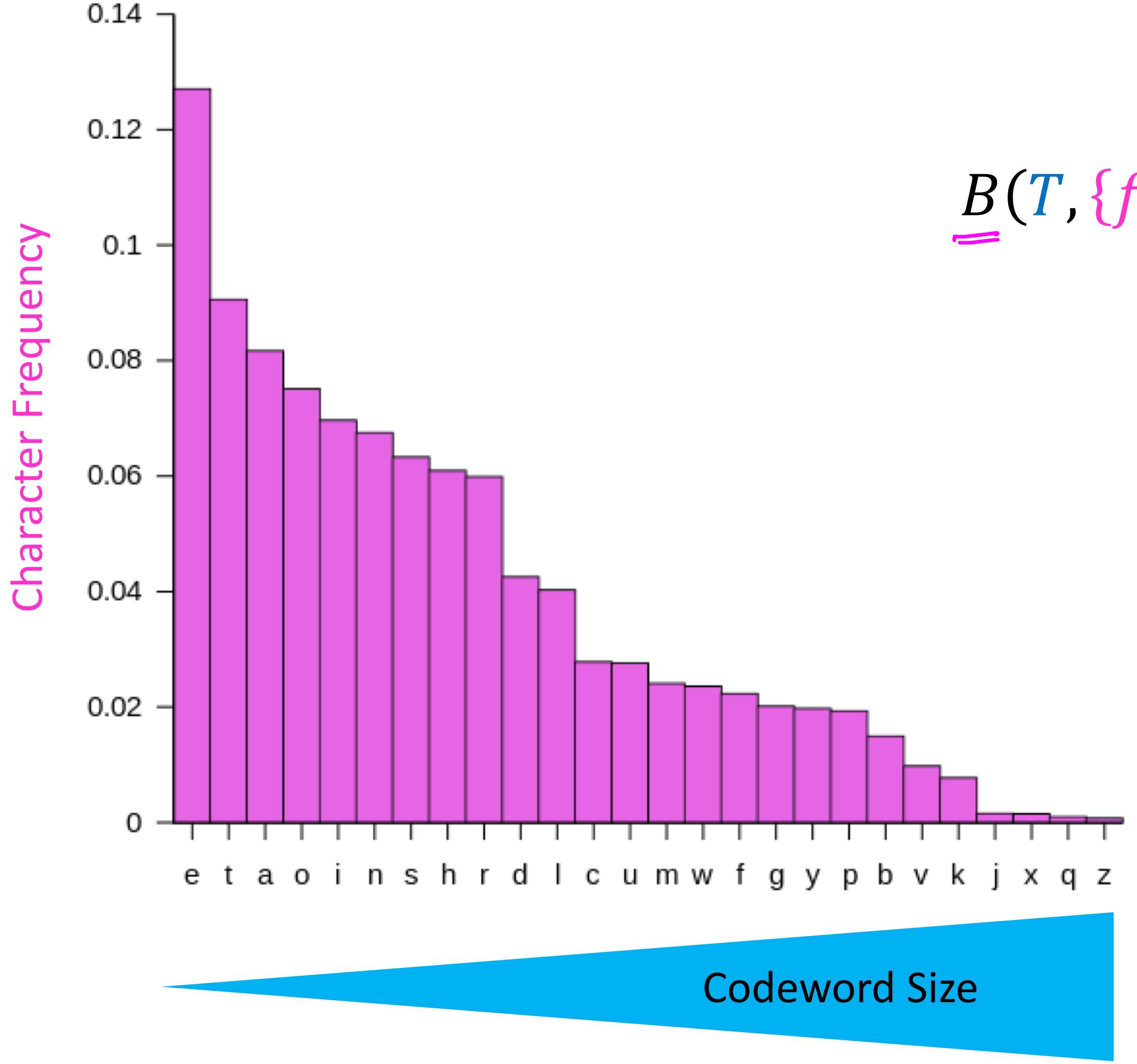
Better Solution: Allow for different characters to have different-size encodings (high frequency → short code)

More efficient coding



$$B(T, \{f_c\}) = \sum_{character\ c} \ell_c f_c$$

More efficient coding

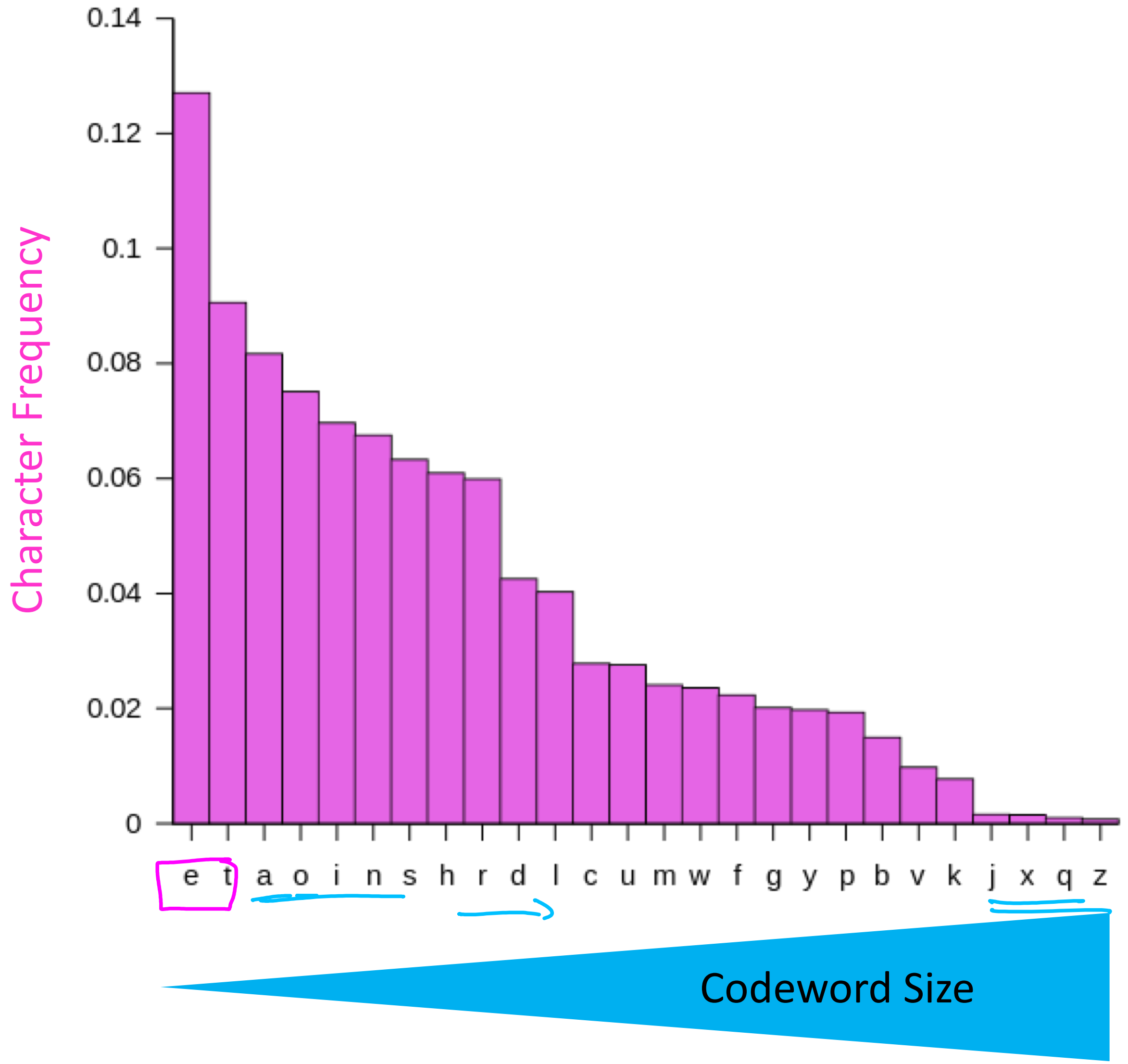


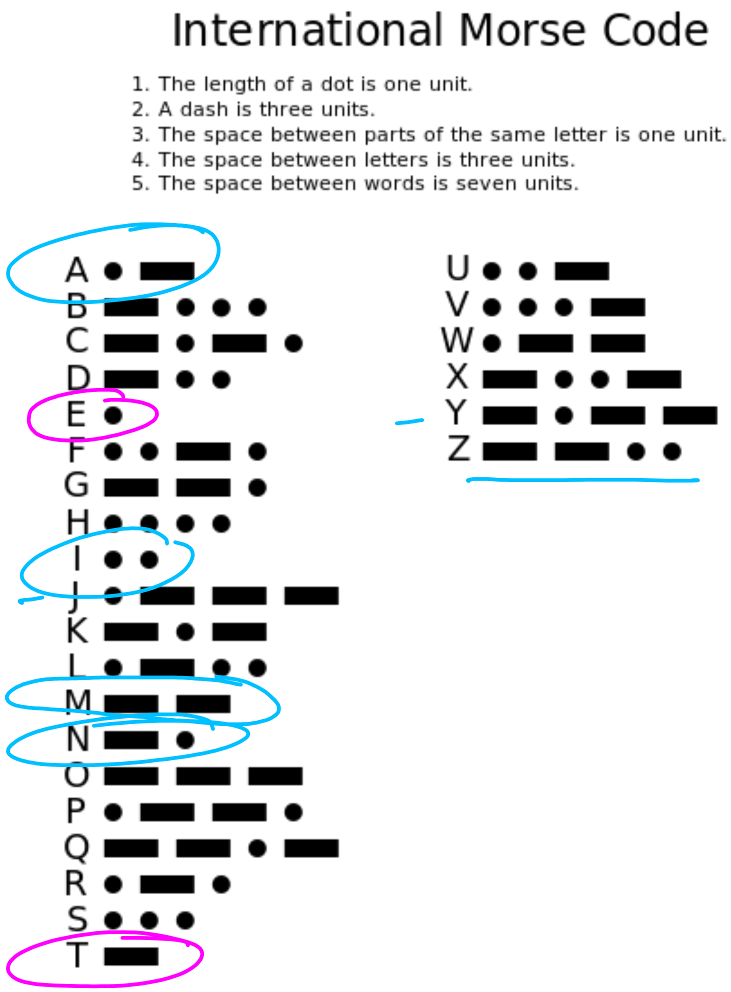
When this is big

$$B(T, \{f_c\}) = \sum_{\text{character } c} \ell_c f_c$$

Make this small

Morse Code





International Morse Code

1. The length of a dot is one unit.
2. A dash is three units.
3. The space between parts of the same letter is one unit.
4. The space between letters is three units.
5. The space between words is seven units.

A	● ▬
B	▬ ● ● ●
C	▬ ● ▬ ●
D	▬ ● ●
E	●
F	● ● ▬ ●
G	▬ ▬ ●
H	● ● ● ●
I	● ●
J	● ▬ ▬ ▬
K	▬ ● ▬
L	● ▬ ● ●
M	▬ ▬
N	▬ ●
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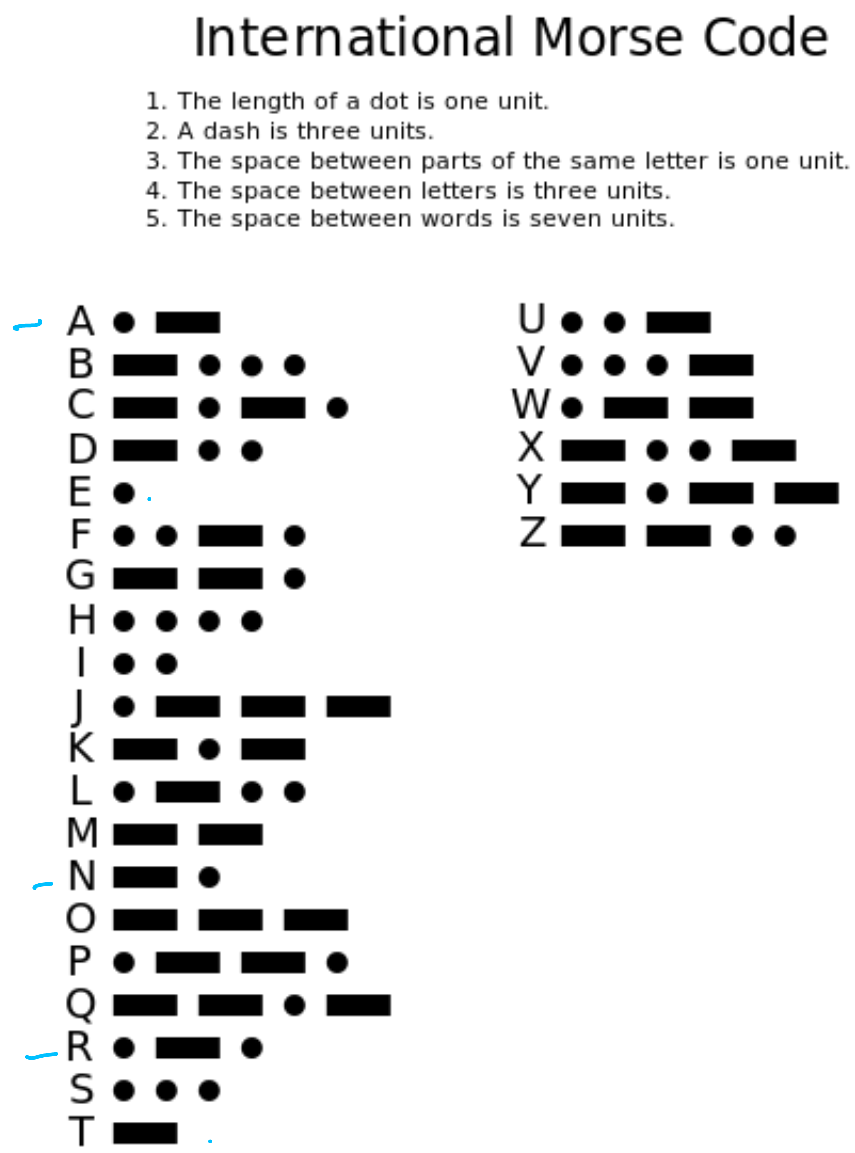
W	● ▬ ▬
X	▬ ● ● ▬
Y	▬ ● ▬ ▬
Z	▬ ▬ ● ●

Problem with Morse Code



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B	▬ ● ● ●
C	▬ ● ▬ ●
D	▬ ● ●
E	● .
F	● ● ▬ ●
G	▬ ▬ ●
H	● ● ● ●
I	● ●
J	● ▬ ▬ ▬
K	▬ ● ▬
L	● ▬ ● ●
M	▬ ▬
– N	▬ ●
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P	● ▬ ▬ ●
Q	▬ ▬ ● ▬
– R	● ▬ ●
S	● ● ●
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U	● ● ▬
V	● ● ● ▬
W	● ▬ ▬
X	▬ ● ● ▬
Y	▬ ● ▬ ▬
Z	▬ ▬ ● ●

Decode: ● ▬ ● ▬

A A

E T E T

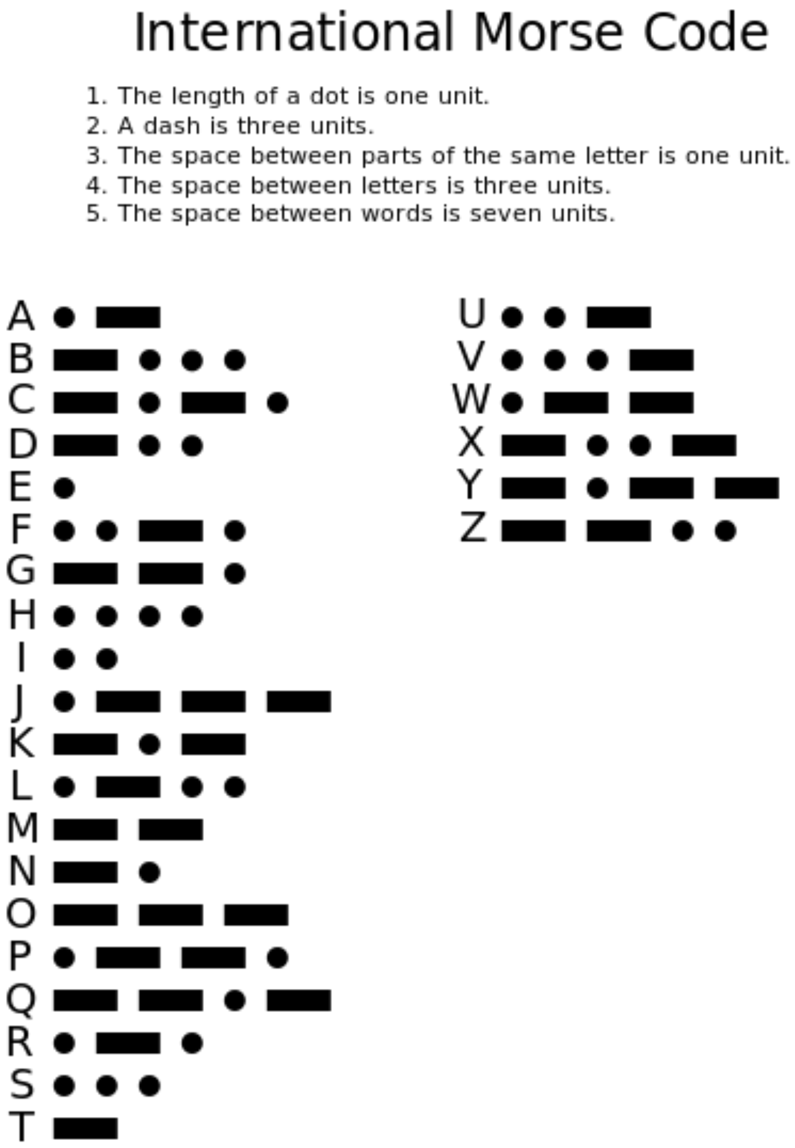
A ET

E T A

R T

E N T

Problem with Morse Code



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C	▬ ● ▬ ●
D	▬ ● ●
E	●
F	● ● ▬ ●
G	▬ ▬ ●
H	● ● ● ●
I	● ●
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N	▬ ●
O	▬ ▬ ▬
P	● ▬ ▬ ●
Q	▬ ▬ ● ▬
R	● ▬ ●
S	● ● ●
T	▬
U	● ● ▬
V	● ● ● ▬
W	● ▬ ▬
X	▬ ● ● ▬
Y	▬ ● ▬ ▬
Z	▬ ▬ ● ●

	A	A
Decode:	● ▬	● ▬
	ET	ET
	R	T
	EN	T

Ambiguous Decoding

Prefix-Free Code

- A prefix-free code is codeword table T such that for any two characters c_1, c_2, if $c_1 \neq c_2$ then $code(c_1)$ is not a prefix of $code(c_2)$

g	0
e	10
l	110
i	1110
w	11110
…	…

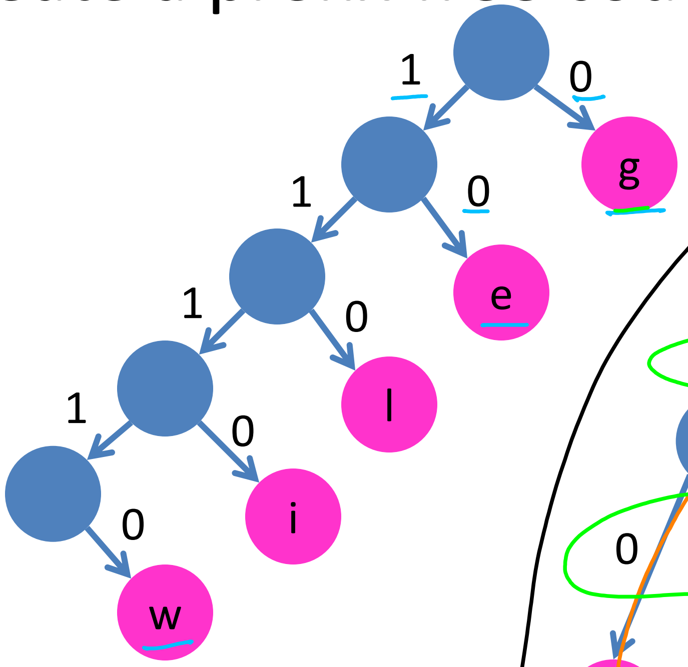
11110|1110|0|011010

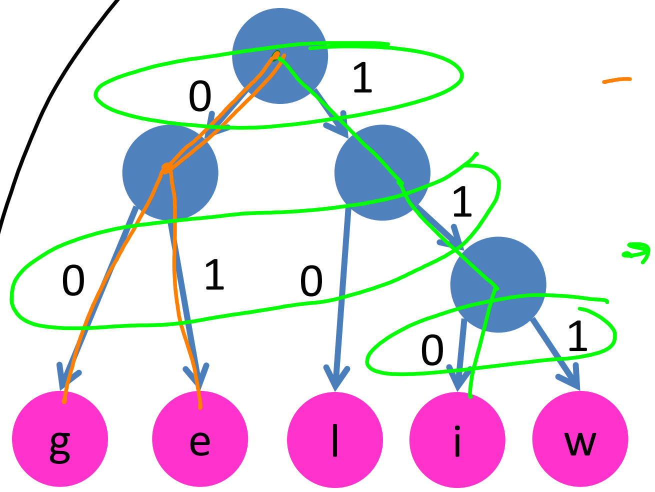
w i gg l e

Binary Trees = Prefix-free Codes

- I can represent any prefix-free code as a binary tree
- I can create a prefix-free code from any binary tree

g	0
e	10
l	110
i	1110
w	11110
…	…





g	00
e	01
l	10
i	110
w	111
…	…

Goal: Shortest Prefix-Free Encoding

- Input: A set of character frequencies $\{f_c\}$
- Output: A prefix-free code T which minimizes

$$B(T, \{f_c\}) = \sum_{character\ c} \ell_c f_c$$

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Huffman Coding!!

Greedy Algorithms

- Require Optimal Substructure
  - – Solution to optimal larger problem contains the optimal solution to a smaller one
  - – Only one subproblem to consider!
- Idea:
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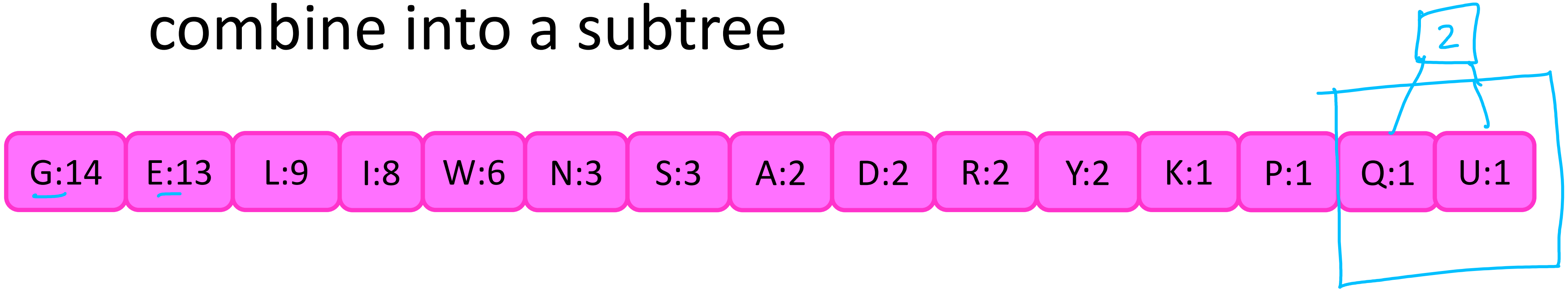
Huffman Algorithm

Greedy choice Property

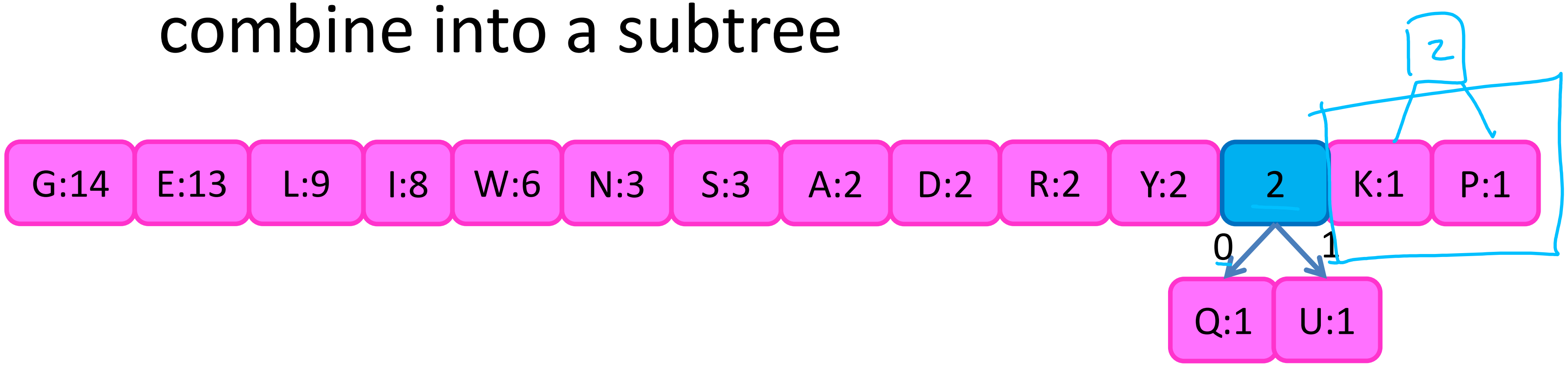
- Choose the least frequent pair, combine into a subtree



G:14 E:13 L:9 I:8 W:6 N:3 S:3 A:2 D:2 R:2 Y:2 K:1 P:1 Q:1 U:1

Huffman Algorithm

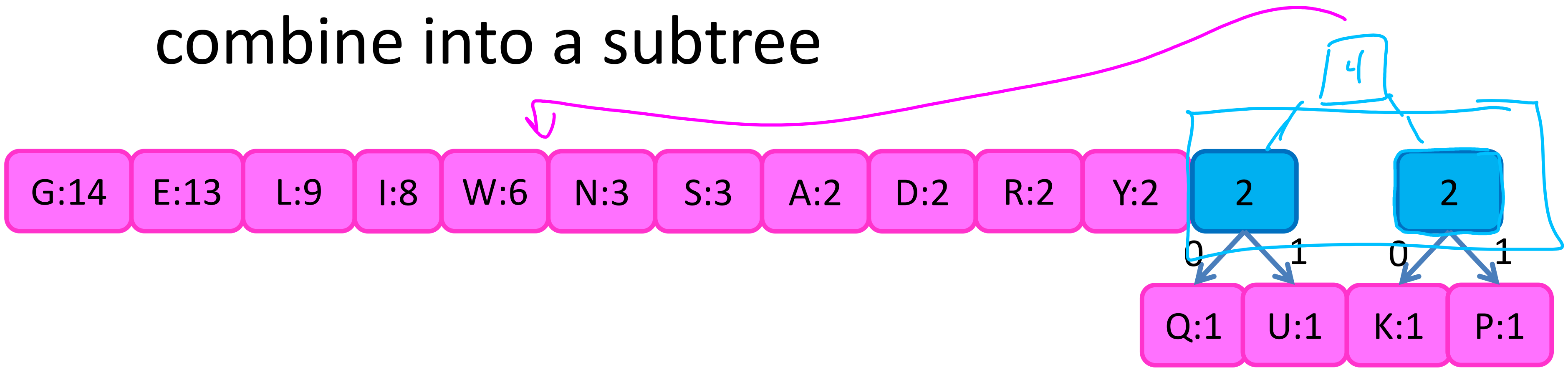
- Choose the least frequent pair, combine into a subtree



Subproblem of size $n-1$!

Huffman Algorithm

- Choose the least frequent pair, combine into a subtree



Huffman Algorithm

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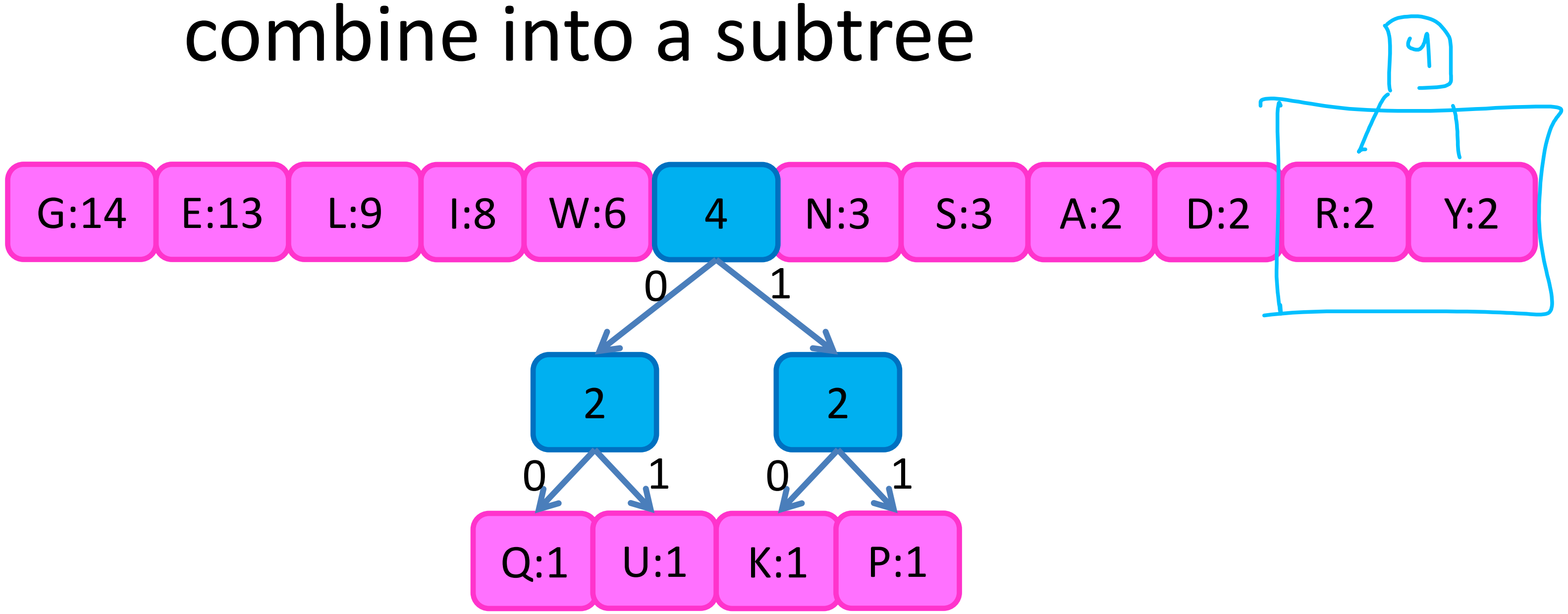

Huffman Algorithm

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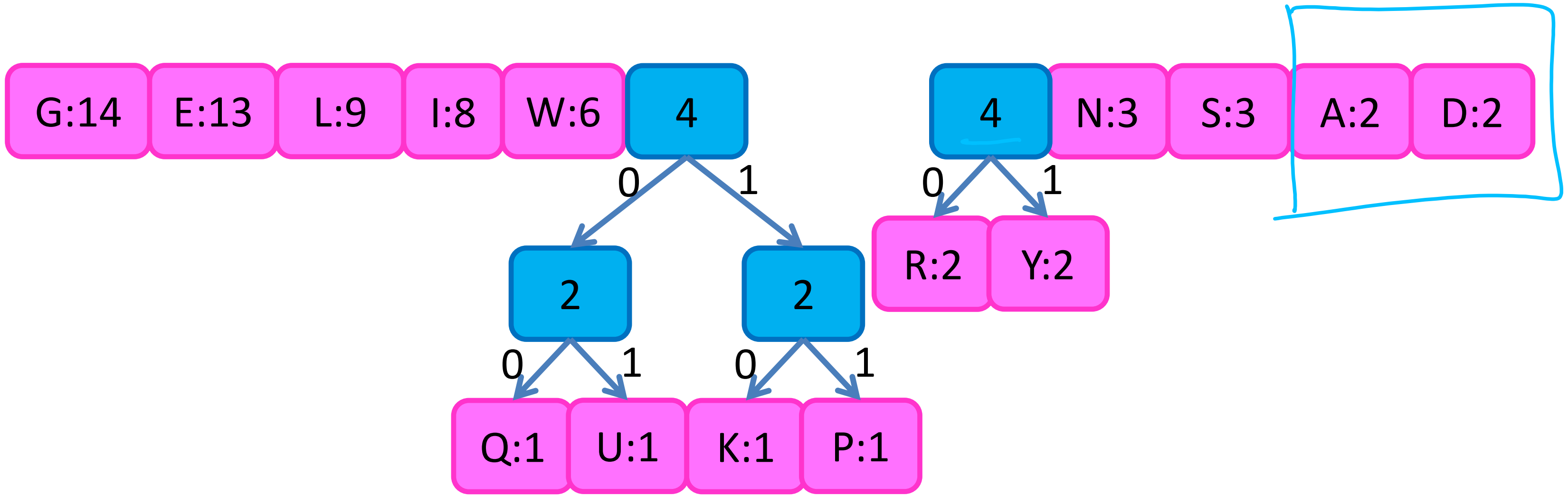

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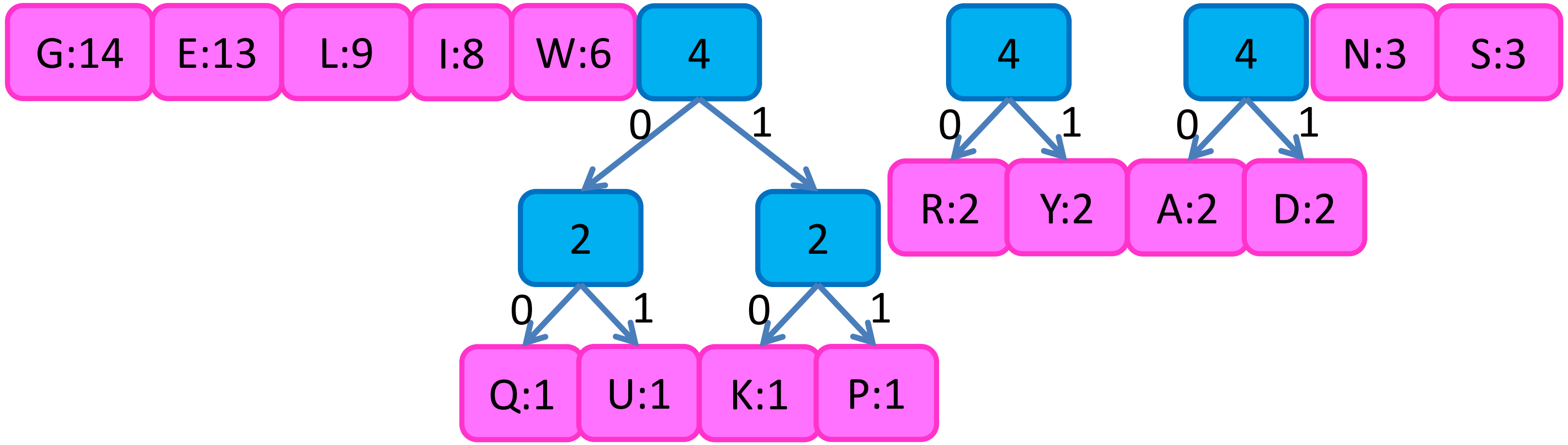
- Choose the least frequent pair, combine into a subtree



Huffman Algorithm

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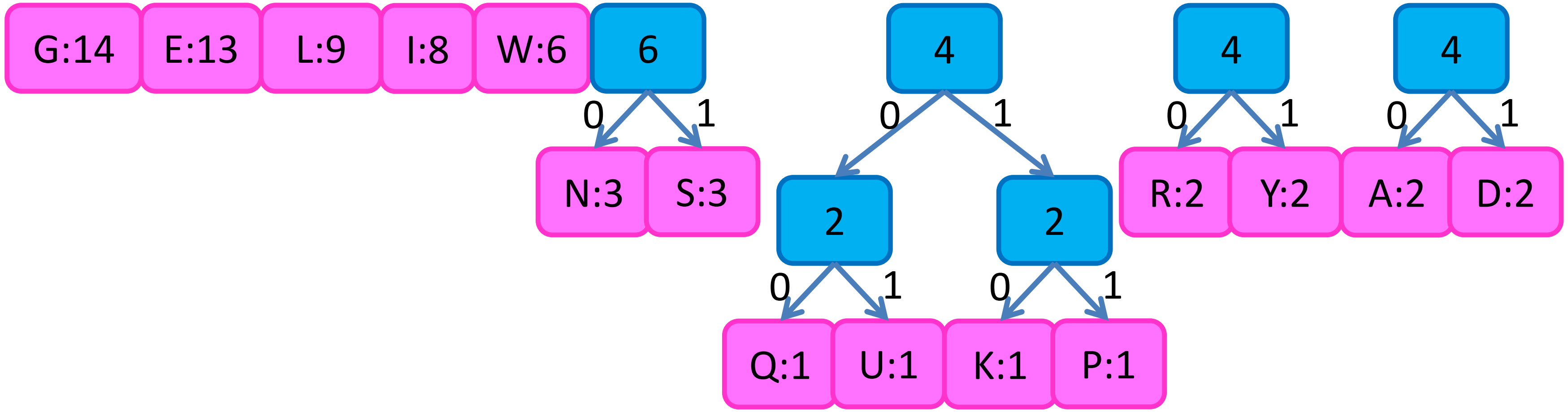

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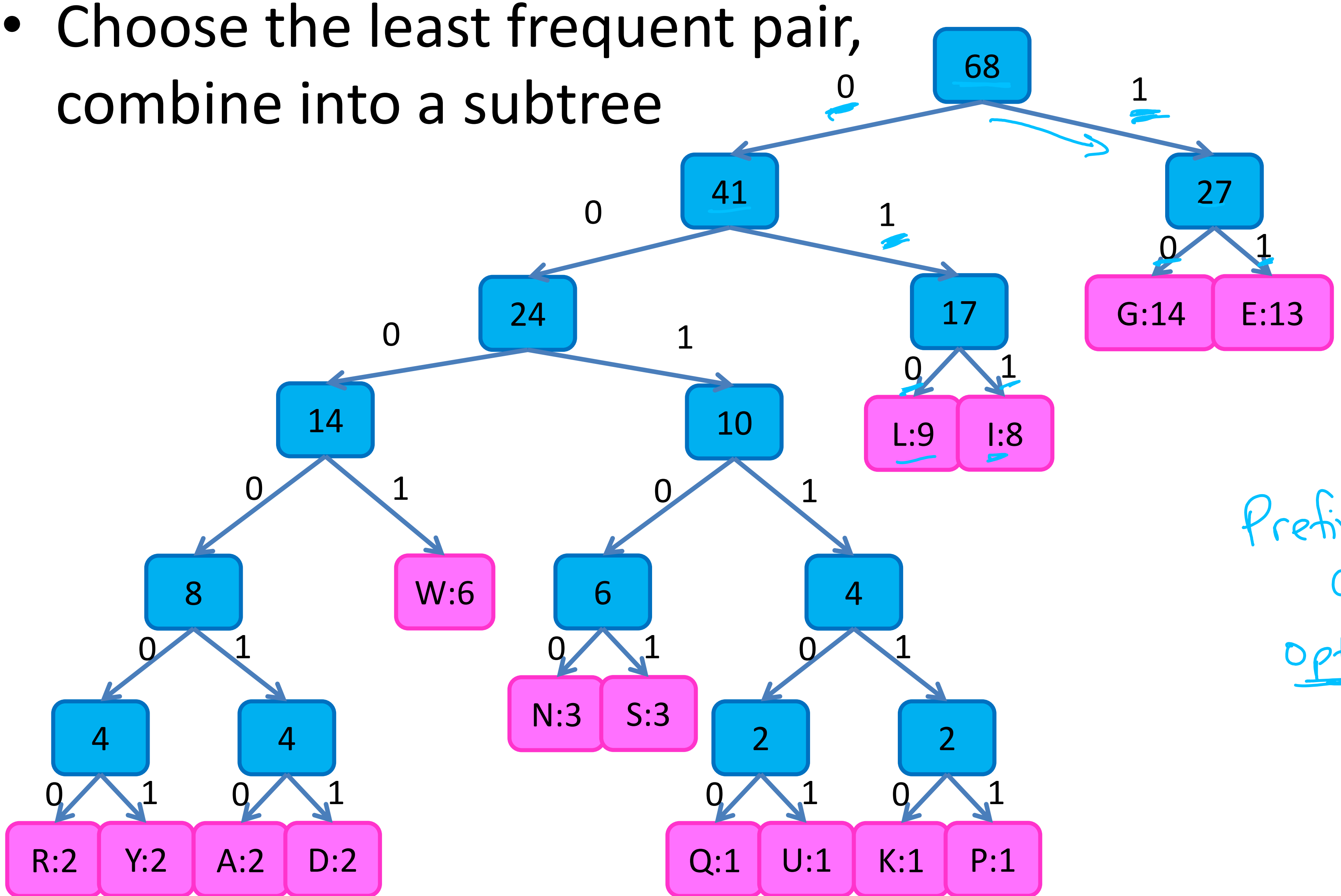


G: 10
E: 11
L: 010
I: 011

Prefix - Free code

Optimal

Exchange argument

- Shows correctness of a greedy algorithm
- Idea:
  - Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
  - How to show my sandwich is at least as good as yours:
    - Show: "I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich"

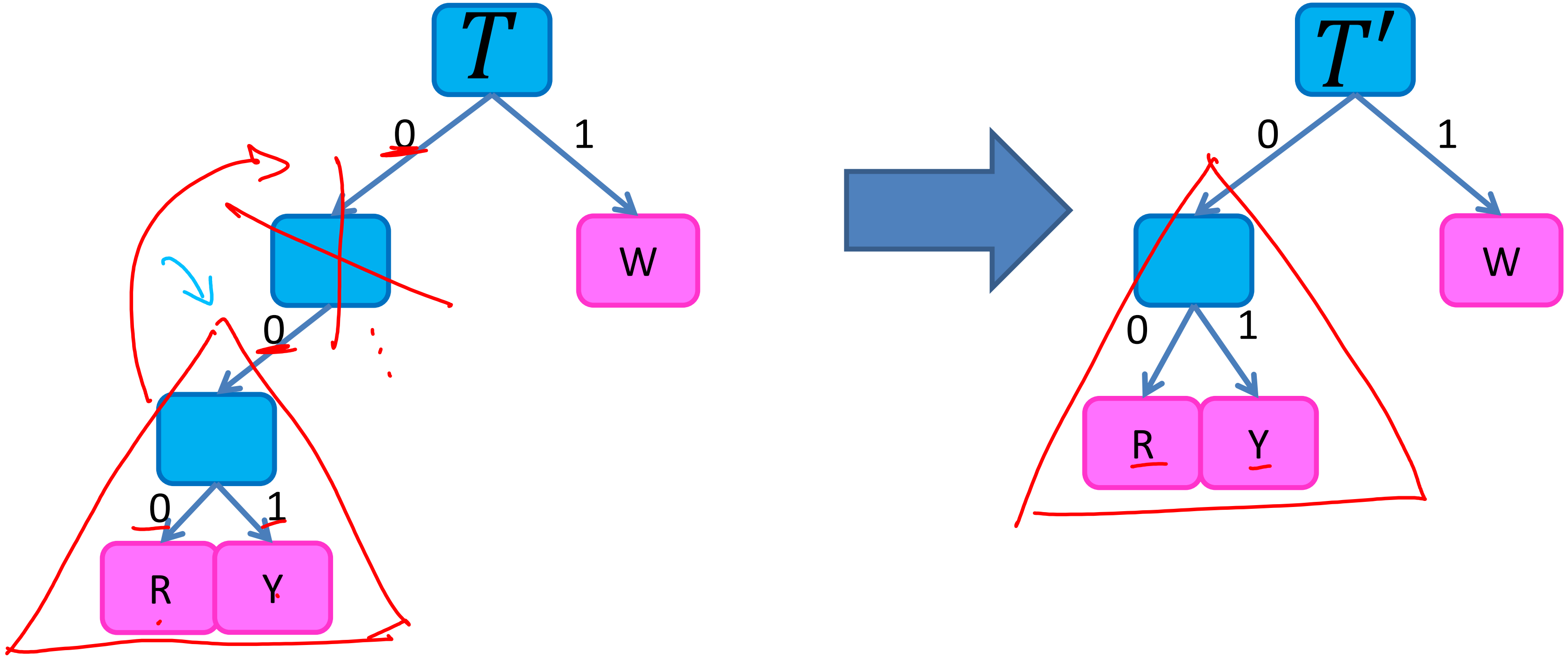
Showing Huffman is Optimal

- Overview:
  - Show that there is **an** optimal tree in which the least frequent characters are siblings – Greedy Choice Property
    - Exchange argument
  - Show that making them siblings and solving the new smaller sub-problem results in **an** optimal solution
    - Proof by contradiction

size n-1

Showing Huffman is Optimal

- First Step: Show any optimal tree is “full” (each node has either 0 or 2 children)



Showing Huffman is Optimal

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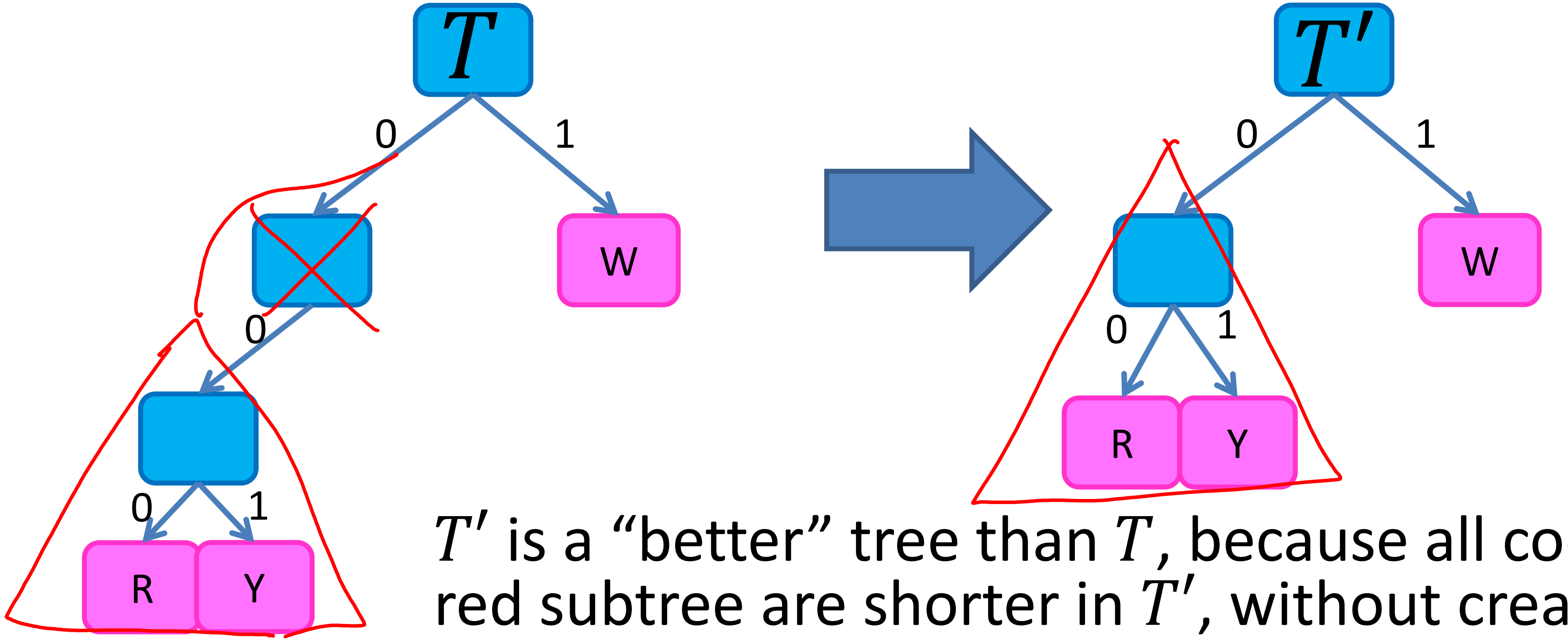
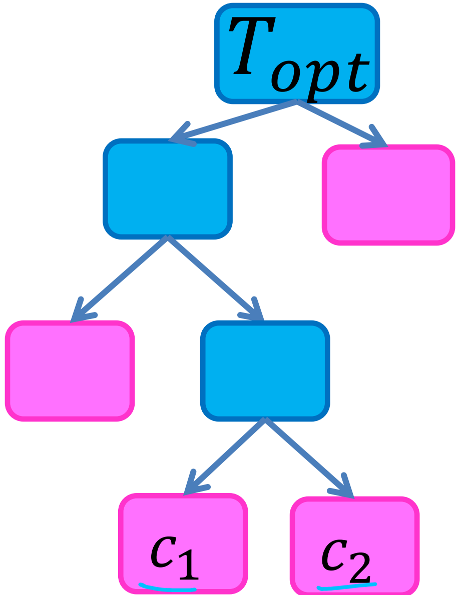


T' is a "better" tree than T, because all codes in red subtree are shorter in T', without creating any longer codes

Huffman Exchange Argument

- Claim: if c_1, c_2 are the least-frequent characters, then there is an optimal prefix-free code s.t. c_1, c_2 are siblings
  - – i.e. codes for c_1, c_2 are the same length and differ only by their last bit

Case 1: Consider some optimal tree T_{opt}. If c_1, c_2 are siblings in this tree, then claim holds



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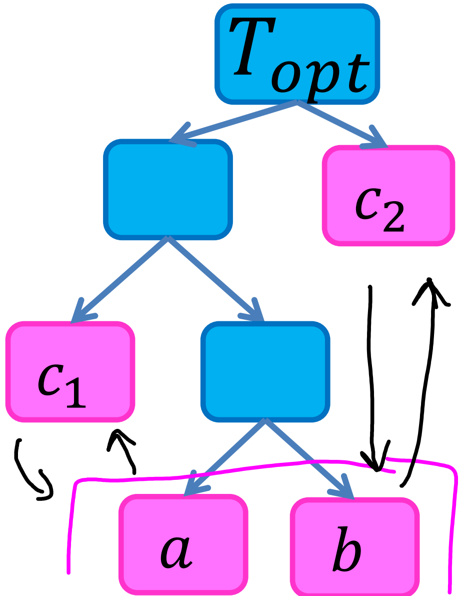
Case 2: Consider some optimal tree T_{opt}, in which c_1, c_2 are not siblings



Let a, b be the two characters of lowest depth that are siblings

(Why must they exist?) – full tree 0 or 2 children

Idea: - show that swap a and c_1, that I do not increase the cost of the tree (encoding).

- repeat similarly with c_2 and b.

Assume: $f_{c_1} \leq f_a$ and $f_{c_2} \leq f_b$

Case 2: c_1, c_2 are not siblings in T_{opt}

- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

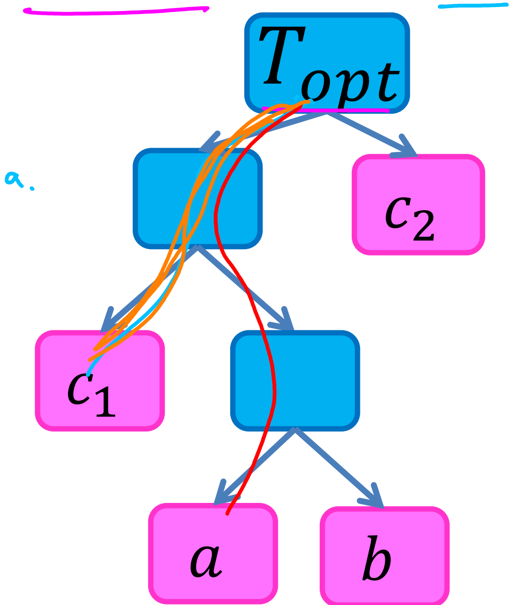
a, b = lowest-depth siblings

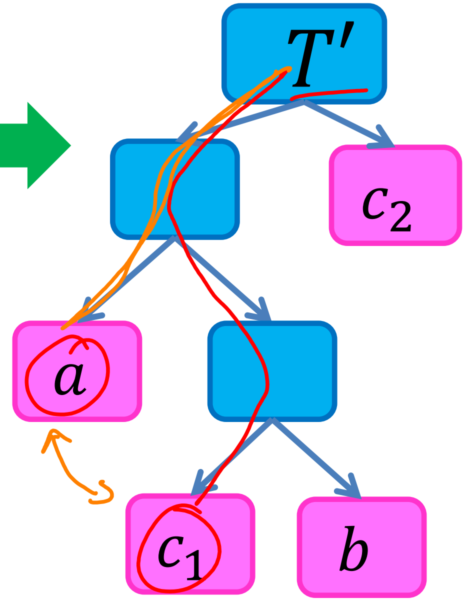
Idea: show that swapping c_1 with a does not increase cost of the tree.
Assume: $f_{c1} \leq f_a$

$$B(T_{opt}) = C + f_{c1}\ell_{c1} + f_a\ell_a$$

$$B(T') = C + f_{c1}\ell_a + f_a\ell_{c1}$$

C = cost to encode all except c_1, a.





Show $B(T_{opt}) \geq B(T')$

$B(T_{opt}) - B(T') \geq 0$

T' is optimal

Case 2: c_1, c_2 are not siblings in T_{opt}

- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

$a, b =$ lowest-depth siblings

Idea: show that swapping c_1 with a does not increase cost of the tree.
Assume: $f_{c1} \leq f_a$

$$B(T_{opt}) = C + f_{c1}\ell_{c1} + f_a\ell_a$$

$$B(T') = C + f_{c1}\ell_a + f_a\ell_{c1}$$

$$B(T_{opt}) - B(T') = C + f_{c_1}\ell_{c_1} + f_a\ell_a - (C + f_{c_1}\ell_a + f_a\ell_{c_1})$$

$$= \cancel{C} + f_{c_1}\ell_{c_1} + f_a\ell_a - \cancel{C} - f_{c_1}\ell_a - f_a\ell_{c_1}$$

$$= \underline{f_{c_1}\ell_{c_1}} + f_a\ell_a - \underline{f_{c_1}\ell_a} - \underline{f_a\ell_{c_1}}$$

$$\overset{-1}{=} f_{c_1}\overset{-1}{(}\ell_{c_1} - \ell_a) + f_a(\ell_a - \ell_{c_1})$$

$$= -f_{c_1}(-\underline{\ell_{c_1} + \ell_a}) + f_a(\underline{\ell_a - \ell_{c_1}})$$

$$= (f_a - f_{c_1})(\ell_a - \ell_{c_1}) \geq 0$$

Case 2: c_1, c_2 are not siblings in T_{opt}

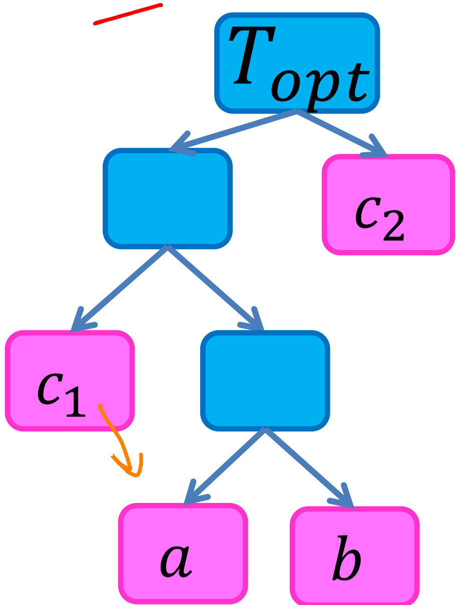
- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

a, b = lowest-depth siblings

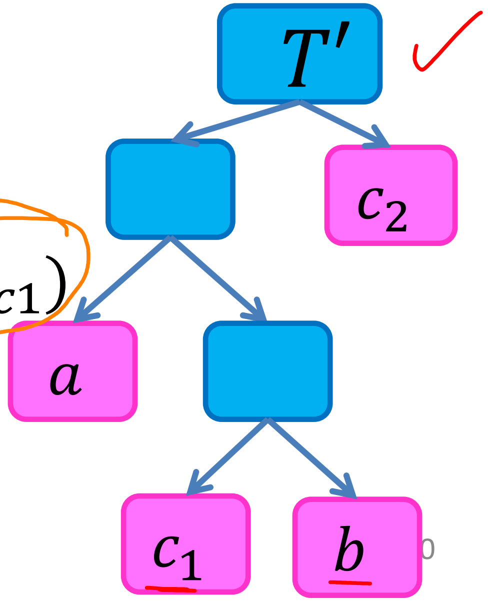
Idea: show that swapping c_1 with a does not increase cost of the tree.

Assume: $f_{c1} \leq f_a$

$$B(T_{opt}) = C + f_{c1}\ell_{c1} + f_a\ell_a$$



$$B(T') = C + f_{c1}\ell_a + f_a\ell_{c1}$$



$$\underbrace{B(T_{opt}) - B(T') = \underbrace{(f_a - f_{c1})}_{\geq 0}\underbrace{(\ell_a - \ell_{c1})}_{\geq 0}}_{\geq 0}$$

$B(T_{opt}) - B(T') \geq 0.$

T' is also optimal.

Case 2:Repeat to swap c_2, b!

- Claim: the least-frequent characters (c_1, c_2), are siblings in some optimal tree

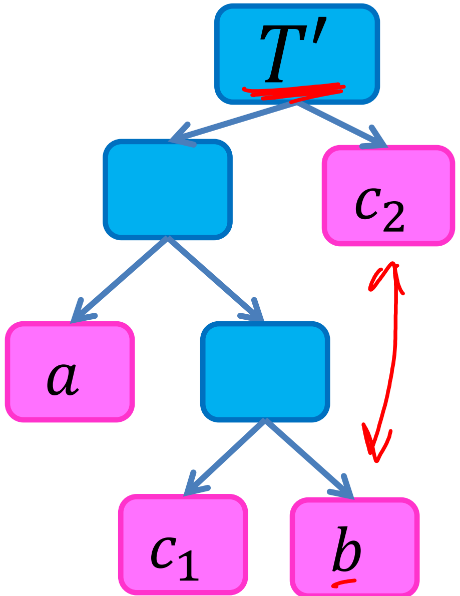
a, b = lowest-depth siblings

Idea: show that swapping c_2 with b does not increase cost of the tree.

Assume: $f_{c2} \leq f_b$

$$B(T') = C + f_{c2}\ell_{c2} + f_b\ell_b$$

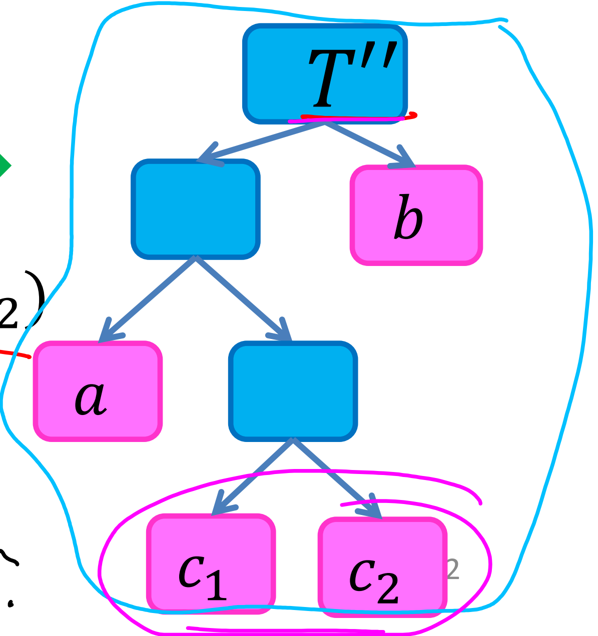
$$B(T'') = C + f_{c2}\ell_b + f_b\ell_{c2}$$



$$B(T') - B(T'') = \underbrace{(f_b - f_{c2})}_{\geq 0}\underbrace{(\ell_b - \ell_{c2})}_{\geq 0}$$

≥ 0

T'' is also optimal. our claim holds.



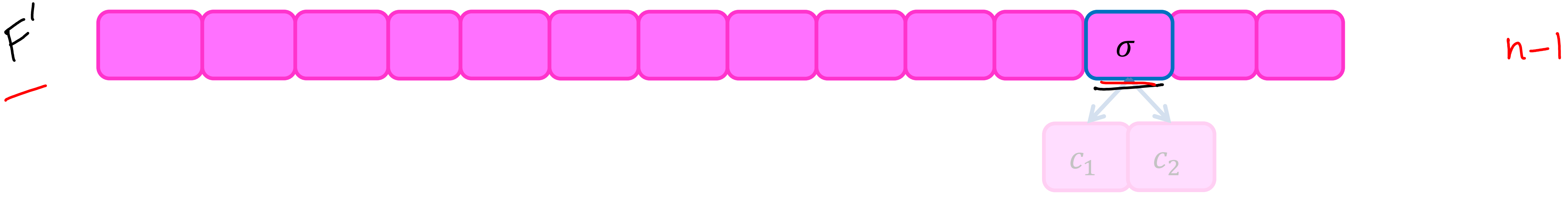
Showing Huffman is Optimal

- Overview:
  - Show that there is an optimal tree in which the least frequent characters are siblings – greedy choice property works ✓
    - Exchange argument
  - Show that making them siblings and solving the new smaller sub-problem results in an optimal solution – optimal substructure works ✓
    - Proof by contradiction

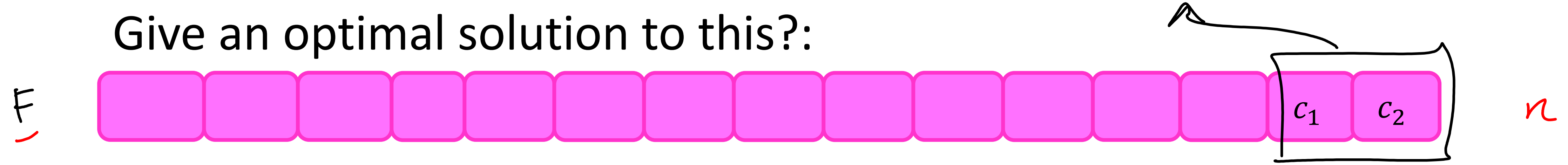
Finishing the Proof

- Show Optimal Substructure
  - Show treating c_1, c_2 as a new "combined" character gives optimal solution

Why does solving this smaller problem:

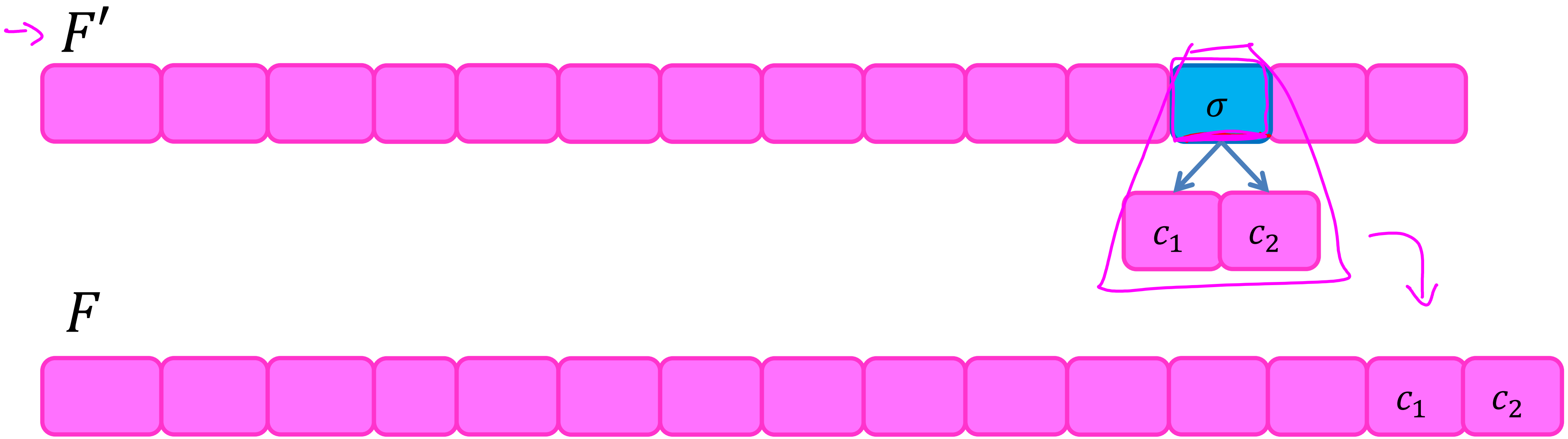


Give an optimal solution to this?:



Optimal Substructure

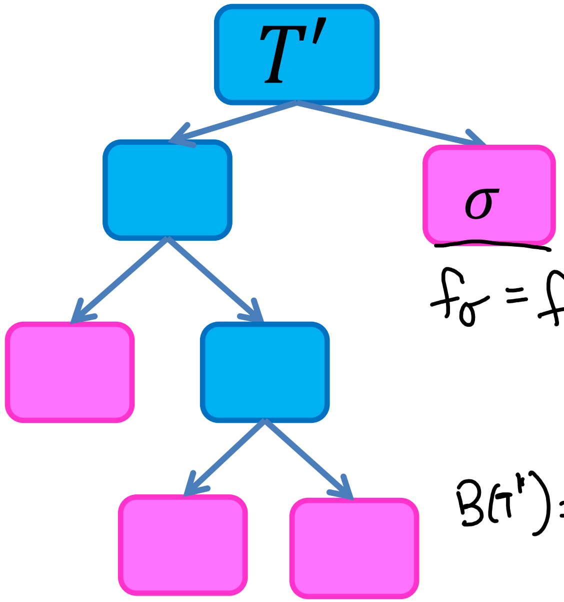
- Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ



Optimal Substructure

- Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ

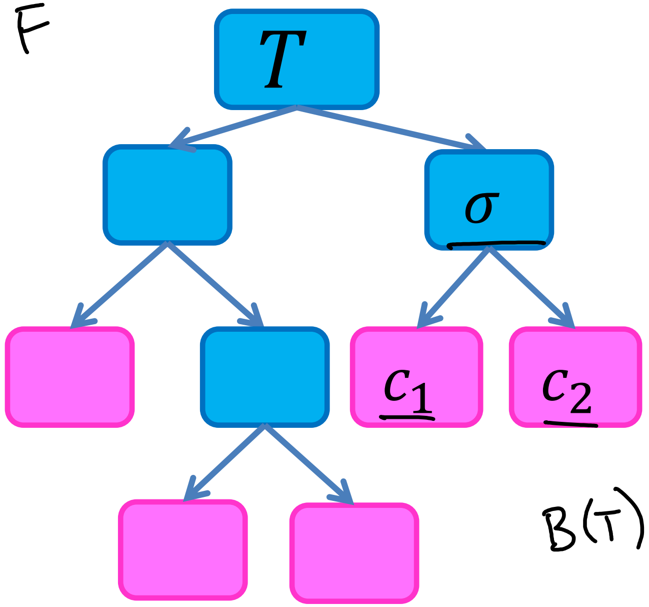
F' If this is optimal



$f_\sigma = f_{c_1} + f_{c_2}$

$B(T') = C + l_\sigma (f_{c_1} + f_{c_2})$

F Then this is optimal



$l_{c_1} = l_\sigma + 1$

$l_{c_2} = l_\sigma + 1$

$B(T) = C + f_{c_1}(l_\sigma + 1) + f_{c_2}(l_\sigma + 1)$

$$\boxed{B(T') = B(T) - f_{c_1} - f_{c_2}}$$

T' – optimal solution for F'

T – solution for F

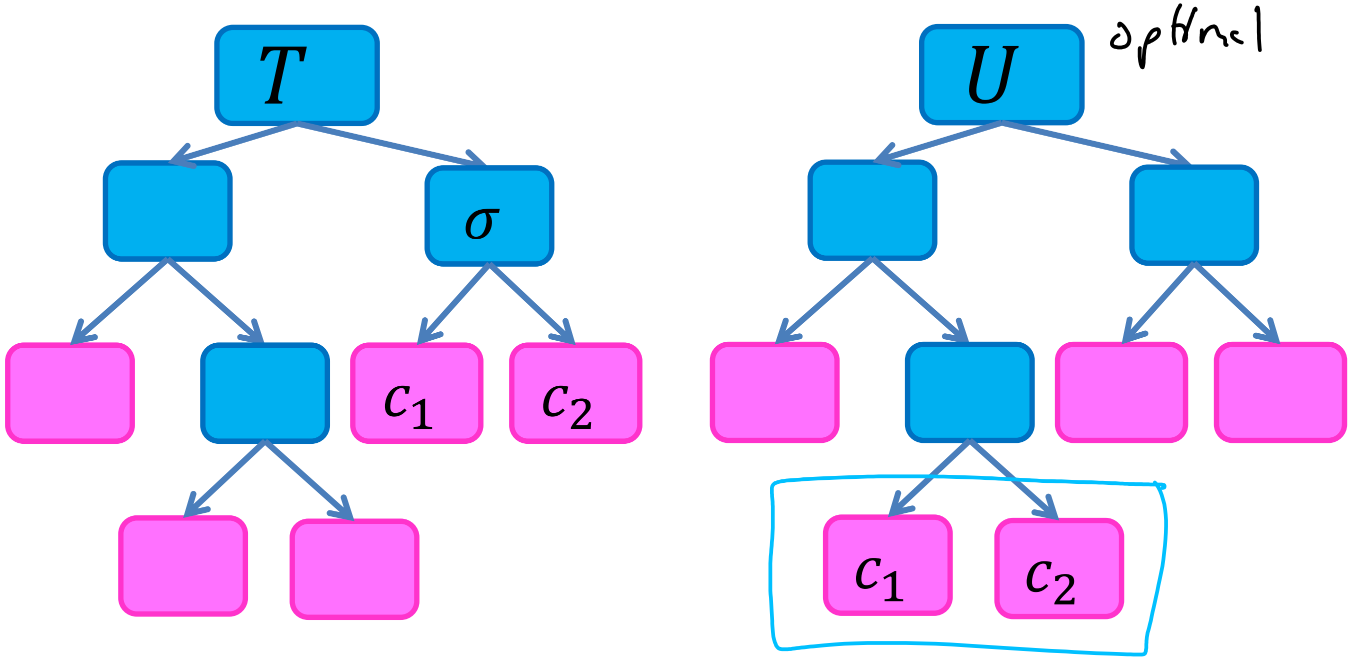
σ → σ, c_1, c_2

Optimal Substructure

- Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ

Suppose toward a contradiction, that T is not optimal. Then let U be a lower-cost tree (U is optimal for F)

$B(U) < B(T)$



Optimal Substructure

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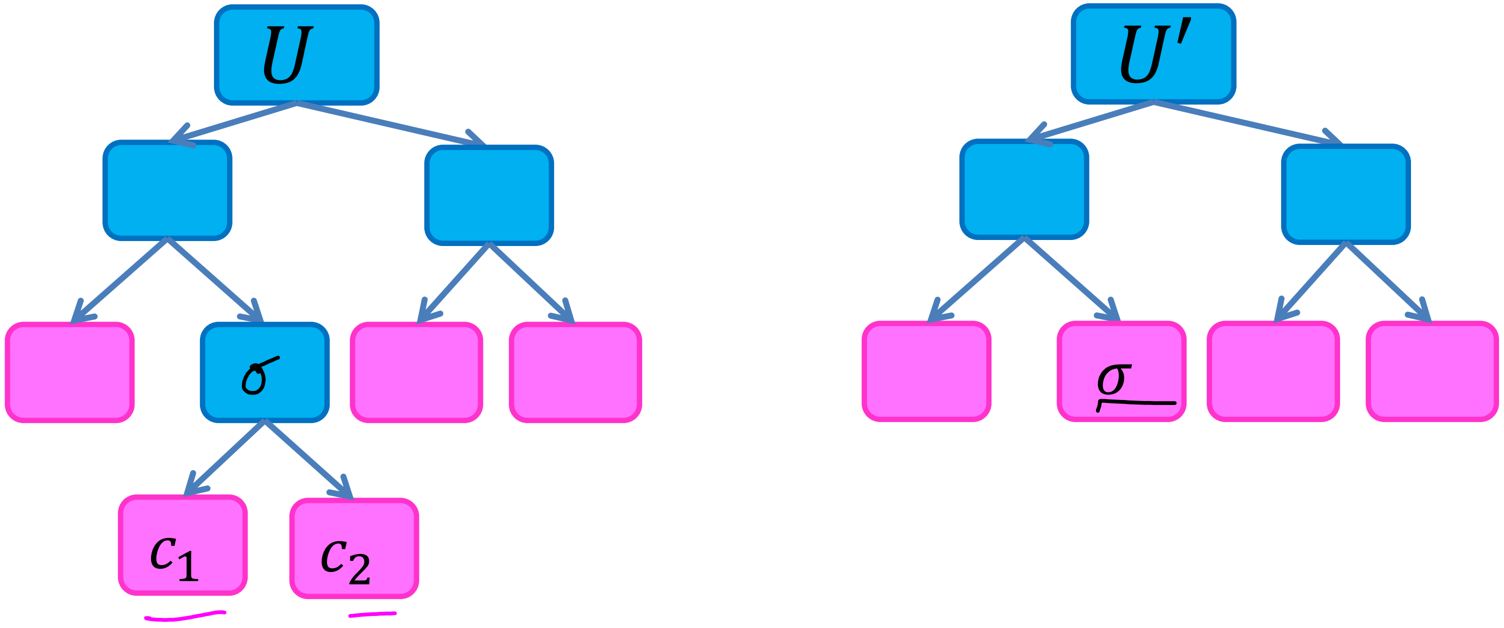


replace $\underset{c_1\ \ c_2}{\boxed{}\!\!\wedge}$ with $\boxed{\sigma}$

creating new solution to F'

know $\underline{B(U) < B(T)}$

$$\underline{B(U')} = \underline{B(U)} - f_{c_1} - f_{c_2}$$

$$< \underline{B(T) - f_{c_1} - f_{c_2}}$$

$$= B(T')$$

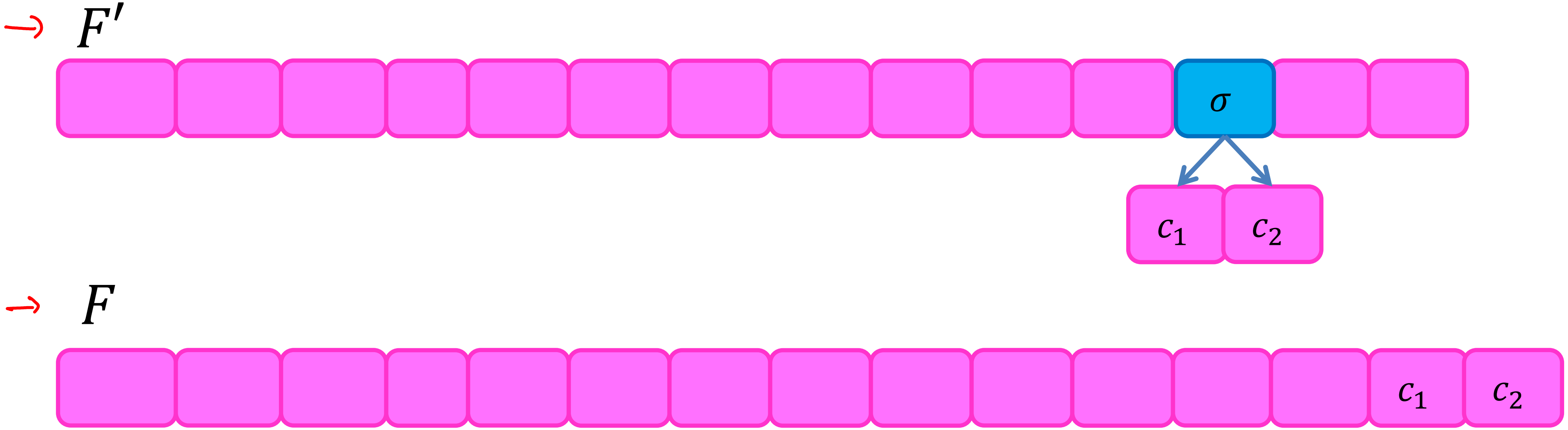
$$\underline{B(U')} < B(T')$$

U' is more optimal than T'? but T' was optimal

Contradiction! Contradicts the optimality of T'. Therefore T is optimal.

Optimal Substructure

- Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ



Optimal Substructure

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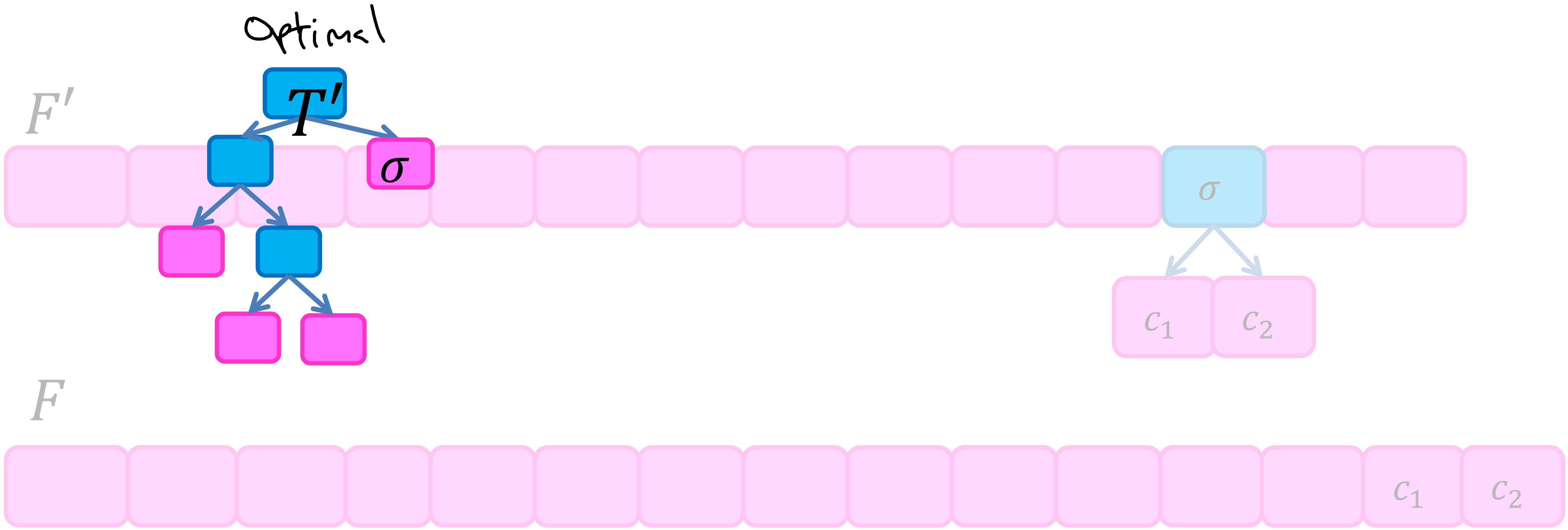

Optimal Substructure

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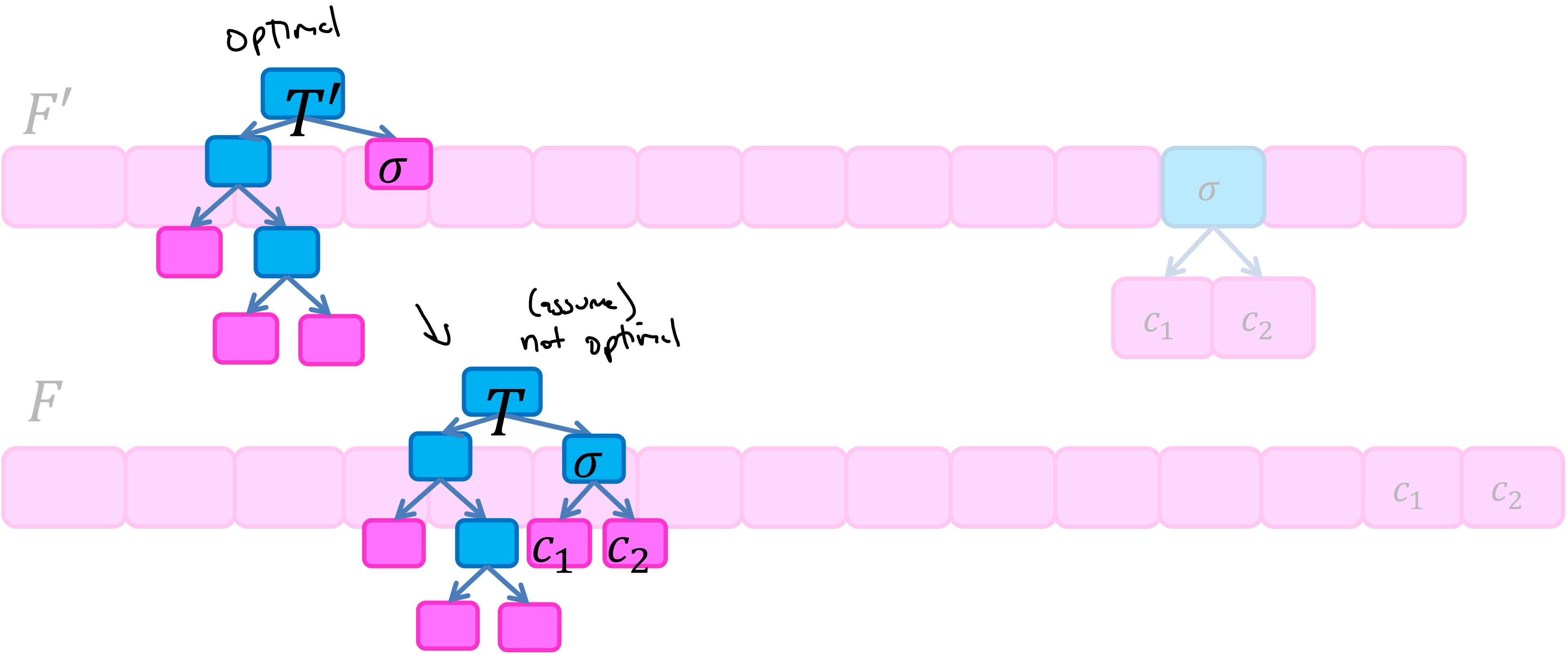

Optimal Substructure

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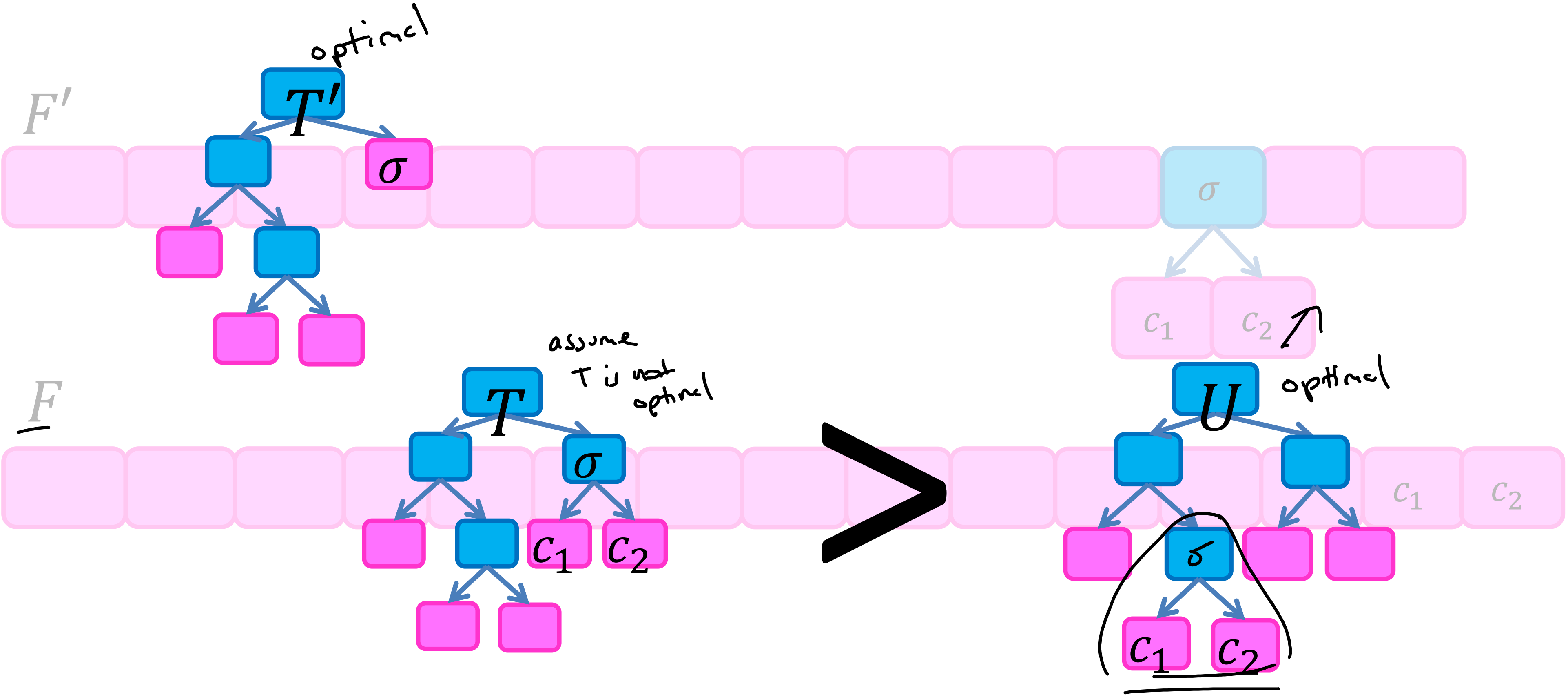
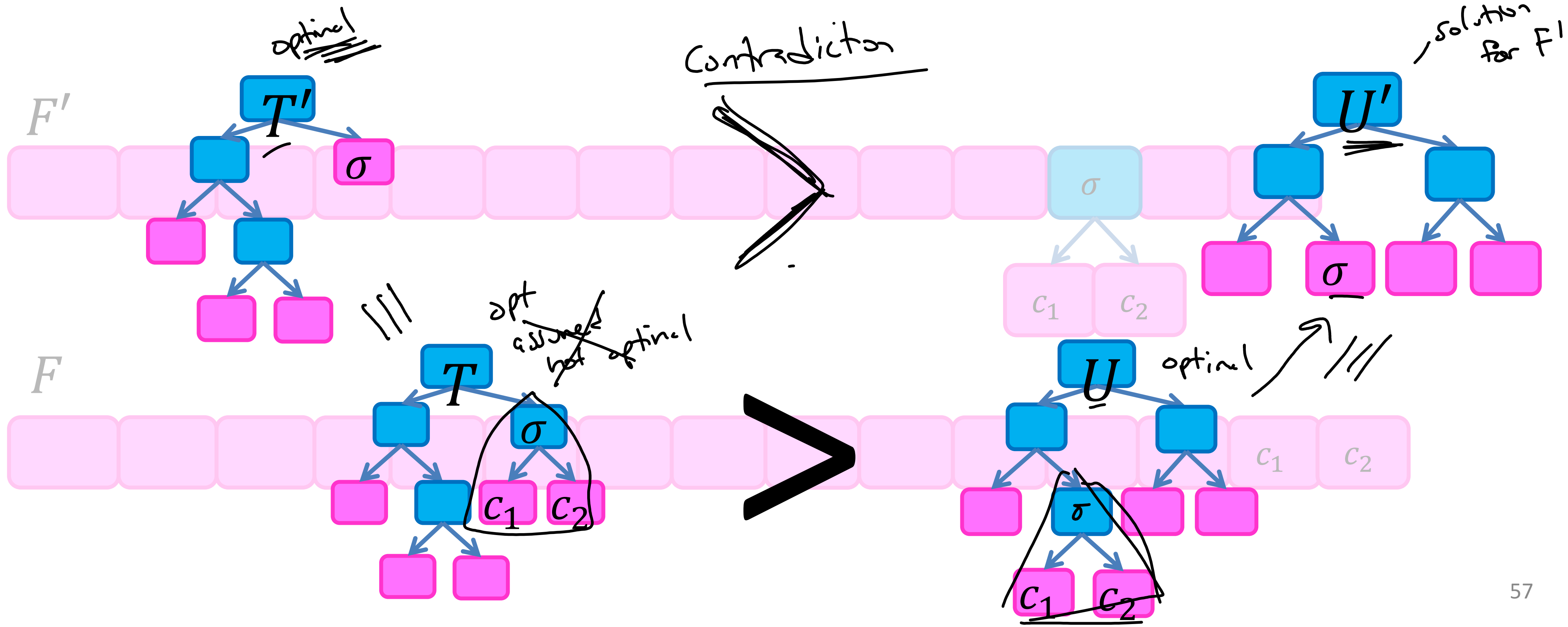


Optimal Substructure

- Claim: An optimal solution for F involves finding an optimal solution for F', then adding c_1, c_2 as children to σ


optimal
F'
T'
σ
Contradiction
solution for F'
U'
σ
σ
c_1
c_2
opt
~~assumed not~~ optimal
F
T
σ
c_1
c_2
optimal
U
c_1
c_2
σ
c_1
c_2