CS4102 Algorithms

Spring 2020

Warm up:

Show that the sum of degrees of all nodes in any undirected graph is even

Show that for any graph $G = (V, E)$, $\sum_{v \in V} \deg(v)$ is even

$\sum_{v \in V} \deg(v)$ is always even

- $\deg(v)$ counts the number of edges incident v
- Consider any edge $\mathrm{e} \in E$
- This edge is incident 2 vertices (on each end)
- This means $2 \cdot |E| = \sum_{v \in V} \deg(v)$
- Therefore $\sum_{v \in V} \deg(v)$ is even

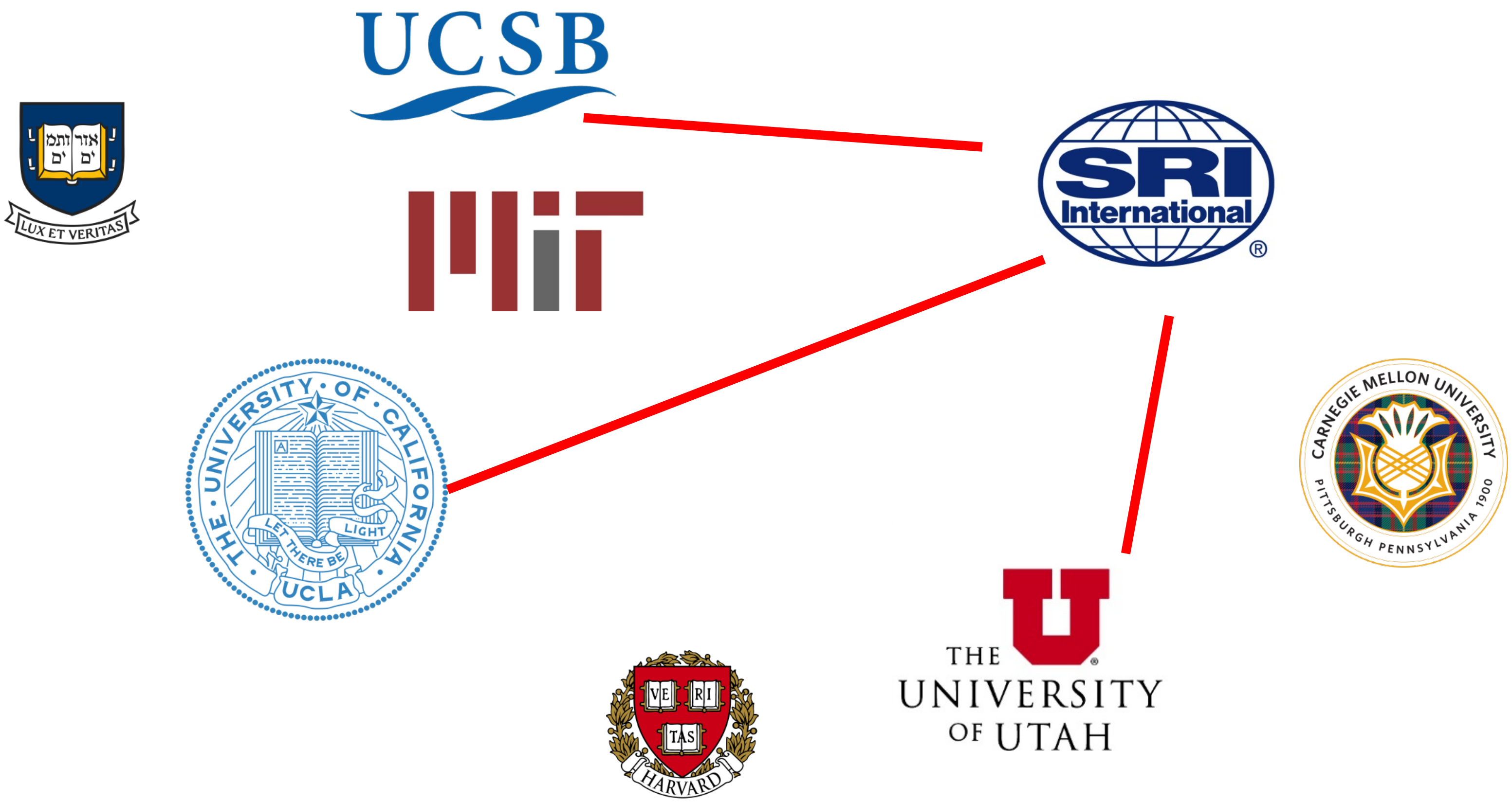
Today's Keywords

- Greedy Algorithms
- Choice Function
- Graphs
- Minimum Spanning Tree
- Kruskal's Algorithm
- Prim's Algorithm
- Cut Theorem

CLRS Readings:

- Chapter 22
- Chapter 23

ARPANET


UCSB
LUX ET VERITAS
SRI International
THE UNIVERSITY OF CALIFORNIA
LET THERE BE LIGHT
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THE UNIVERSITY OF UTAH
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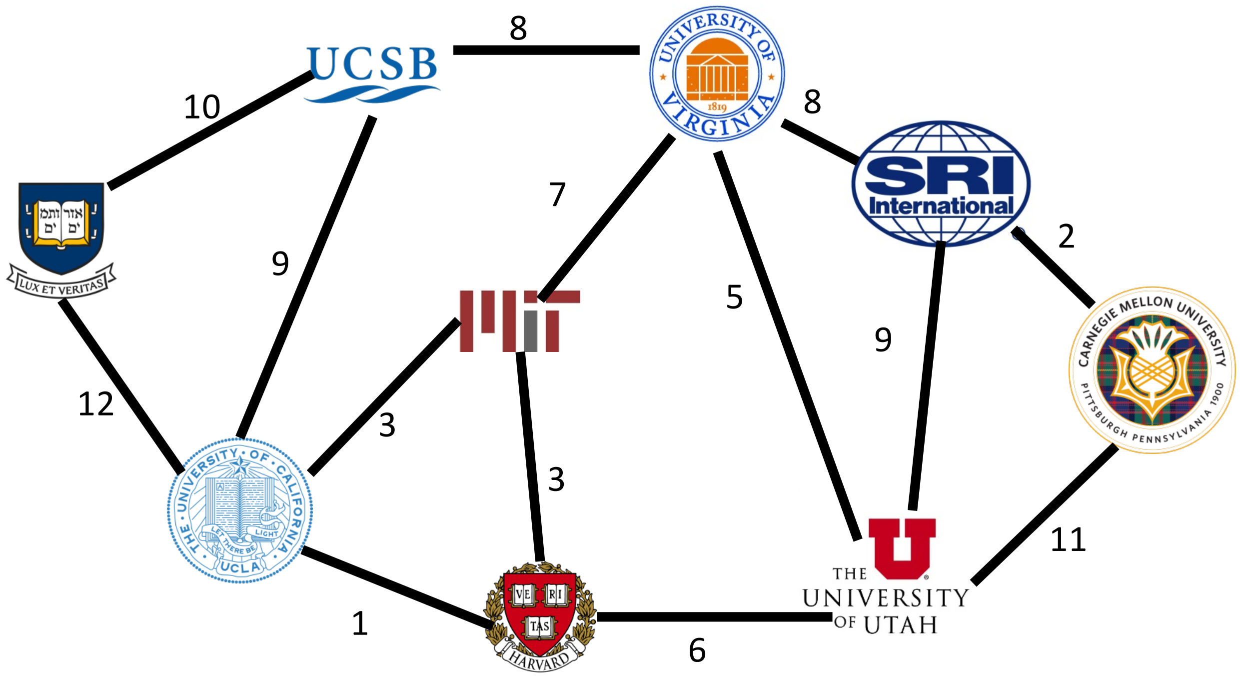
Problem



Find a
Minimum
Spanning Tree

We need to connect together all these places into a network
We have feasible wires to run, plus the cost of each wire
Find the cheapest set of wires to run to connect all places

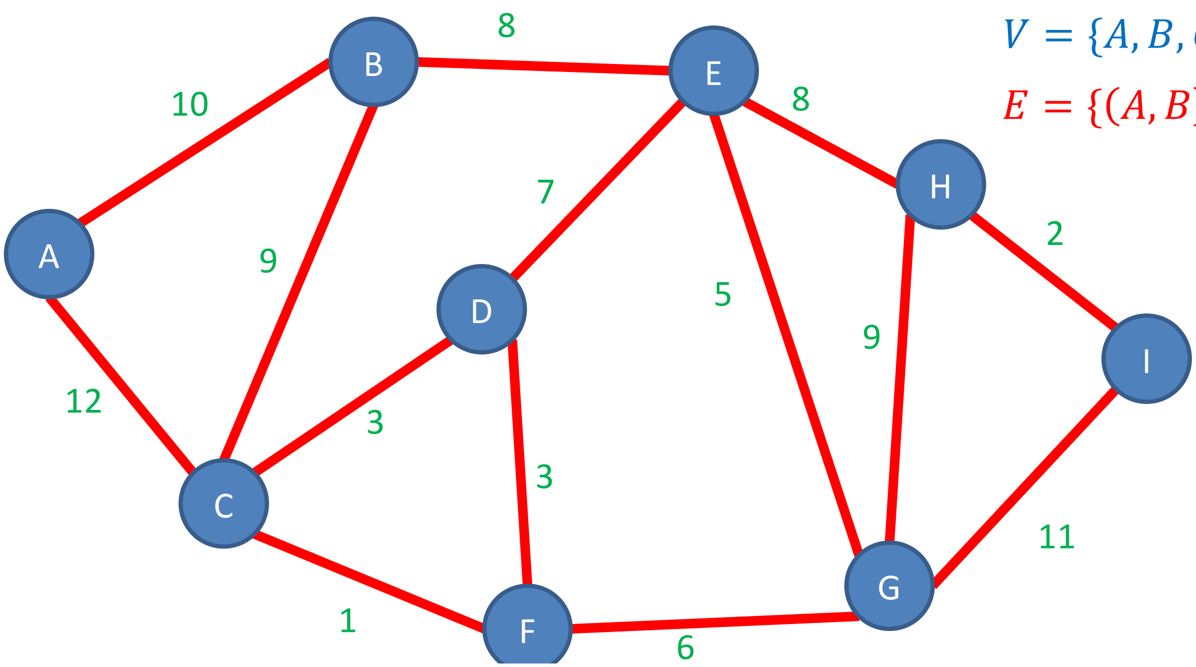
Graphs

Vertices/Nodes

Definition: $G = (V, E)$

Edges

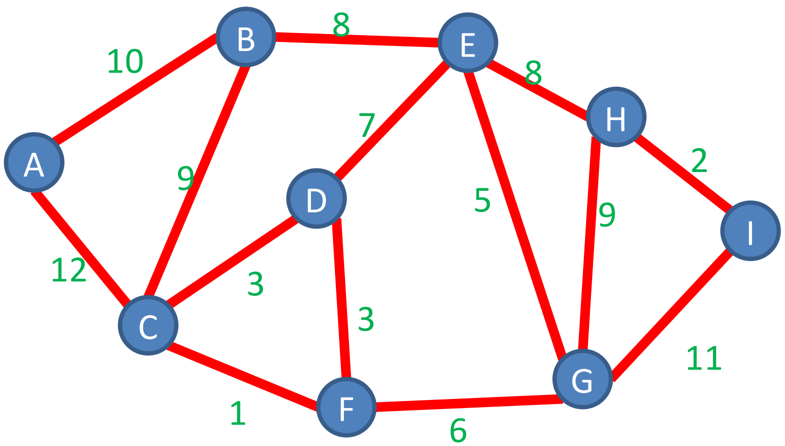
$w(e)$ = weight of edge e



$V = \{A, B, C, D, E, F, G, H, I\}$

$E = \{(A, B), (A, C), (B, C), \dots\}$

Adjacency List Representation



A	B	C		
B	A	C	E	
C	A	B	D	F
D	C	E	F	
E	B	D	G	H
F	C	D	G	
G	E	F	H	I
H	E	G	I	
I	G	H		

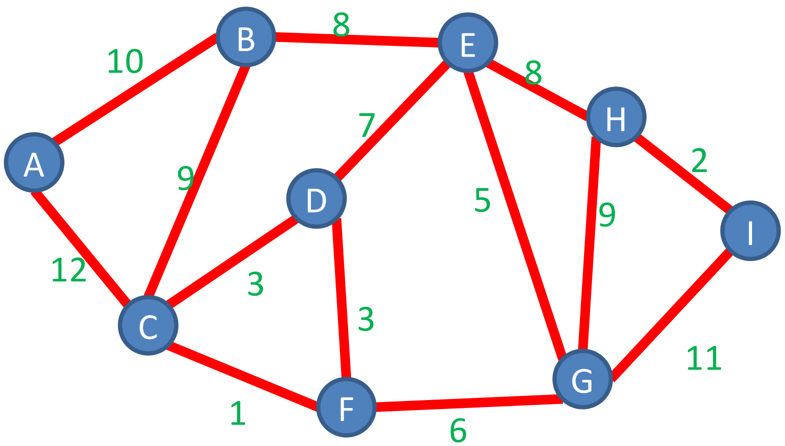
<u>Tradeoffs</u>

Space: $V + E$

Time to list neighbors: $Degree(A)$

Time to check edge (A, B): $Degree(A)$

Adjacency Matrix Representation



	A	B	C	D	E	F	G	H	I
A		1	1						
B	1		1		1				
C	1	1		1		1			
D			1		1	1			
E		1		1			1	1	
F			1	1			1		
G					1	1		1	1
H					1		1		1
I							1	1	

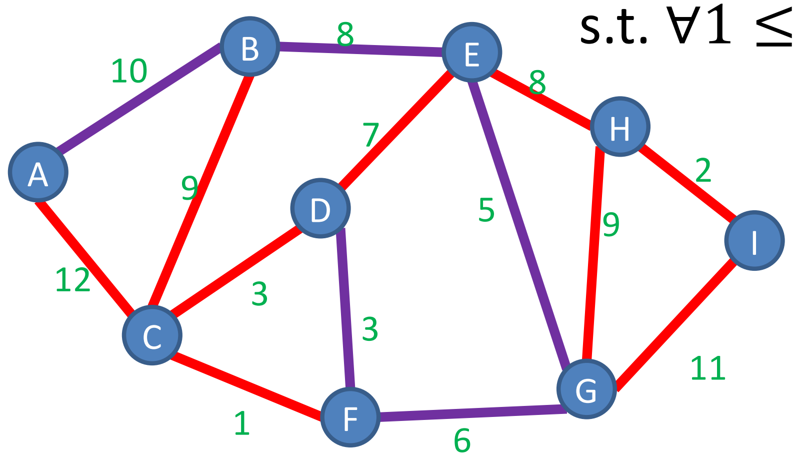
<u>Tradeoffs</u>

Space: V^2

Time to list neighbors: V

Time to check edge (A, B):$O(1)$

Definition: Path



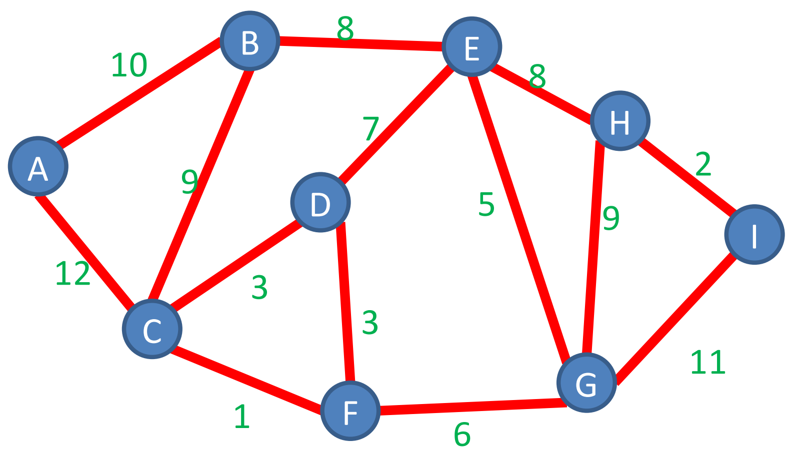
A sequence of nodes $(v_1, v_2, \ldots, v_k)$
s.t. $\forall 1 \le i \le k-1$, $(v_i, v_{i+1}) \in E$

<u>Simple Path:</u>
A path in which each node appears at most once

<u>Cycle:</u>
A path of > 2 nodes in which $v_1 = v_k$ and all other nodes appear at most once

Definition: Connected Graph

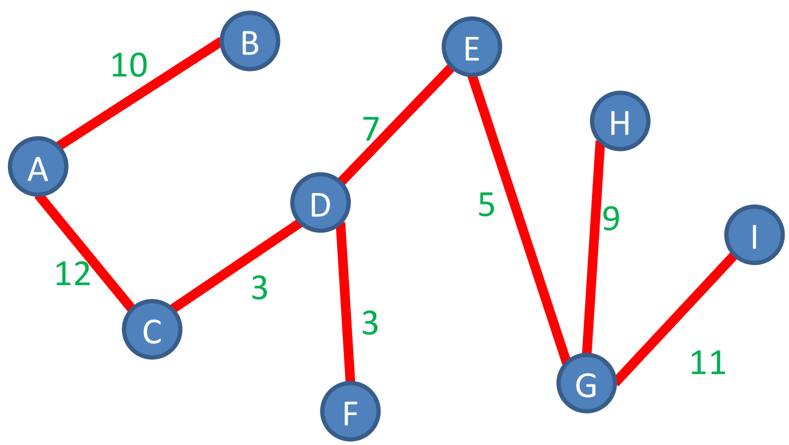
A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from v_1 to v_2



Note: we're talking about undirected graphs here.
It's a more complex situation for directed graphs (see "strongly connected").

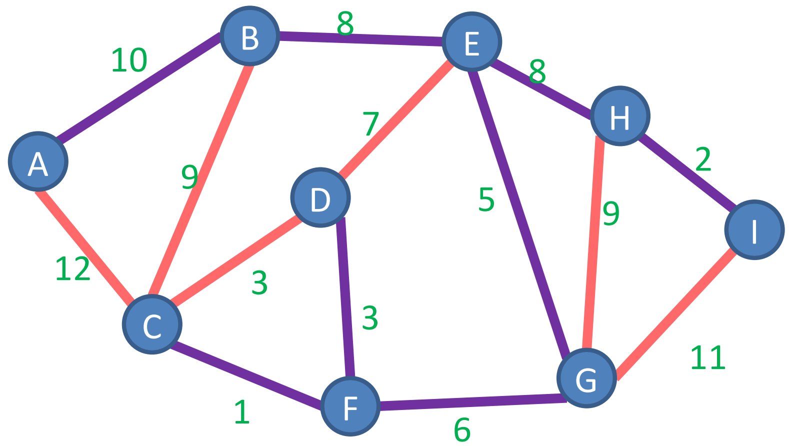
Definition: Tree

A connected graph with no cycles



Definition: Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$



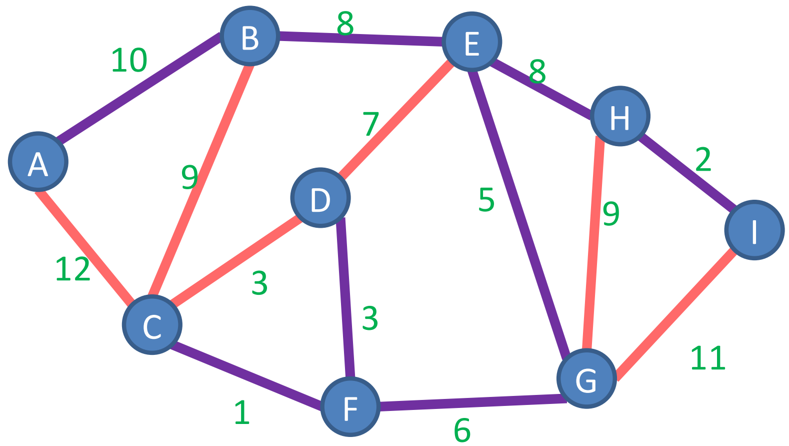
How many edges does T have?

$V - 1$

Definition: Minimum Spanning Tree

A Tree $T = (V_T, E_T)$ which connects ("spans") all the nodes in a graph $G = (V, E)$, that has minimal cost

$$Cost(T) = \sum_{e \in E_T} w(e)$$



How many edges does T have?

$V - 1$

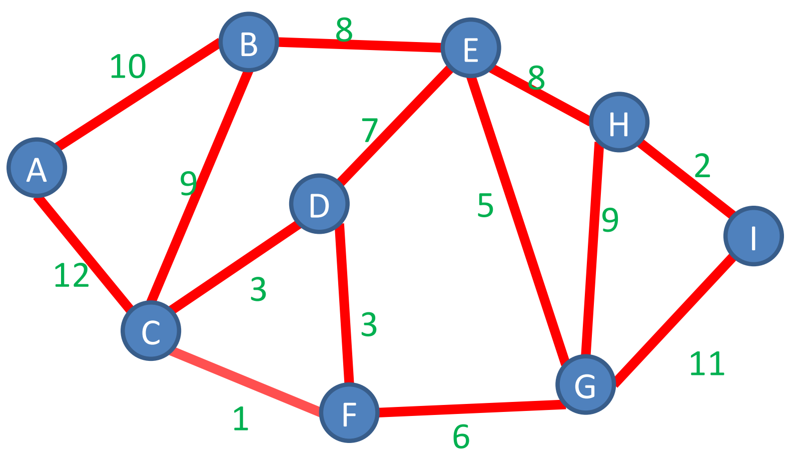
Greedy Algorithms

- Require Optimal Substructure
  - Optimal solution to a problem contains optimal solutions to subproblems
  - Only one subproblem to consider!
- Idea:
  1. Identify a greedy choice property
     - How to make a choice guaranteed to be included in some optimal solution
  2. Repeatedly apply the choice property until no subproblems remain

Kruskal's Algorithm

We want a tree, but we'll start an empty forest A (set of trees). A will eventually become just one tree.

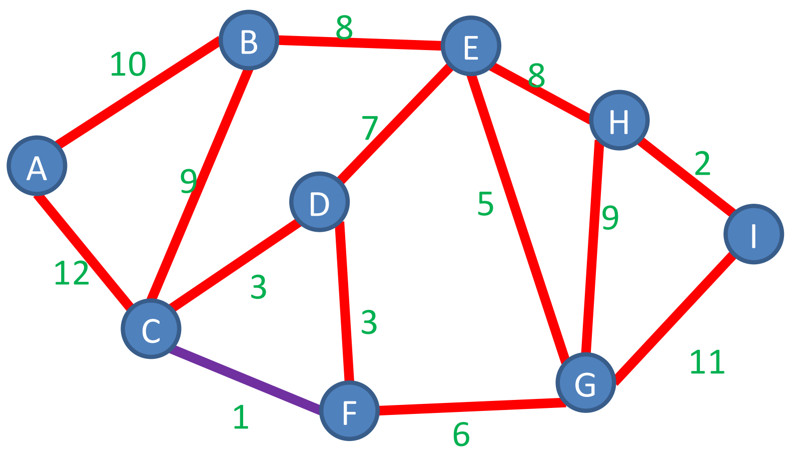
Add to A the lowest-weight edge that does not create a cycle



Kruskal's Algorithm

Start with an empty forest A

Add to A the lowest-weight edge that does not create a cycle.

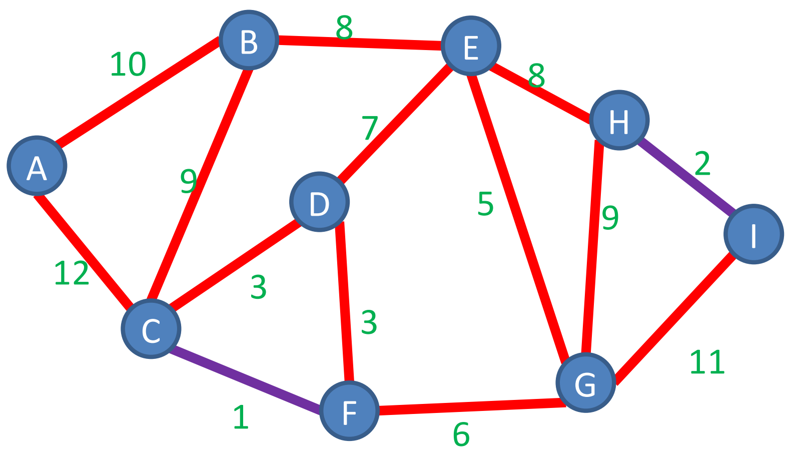


Question: what organization of info about the graph do we need to be able to add an edge?

Kruskal's Algorithm

Start with an empty forest A

Add to A the lowest-weight edge that does not create a cycle



Kruskal's Algorithm

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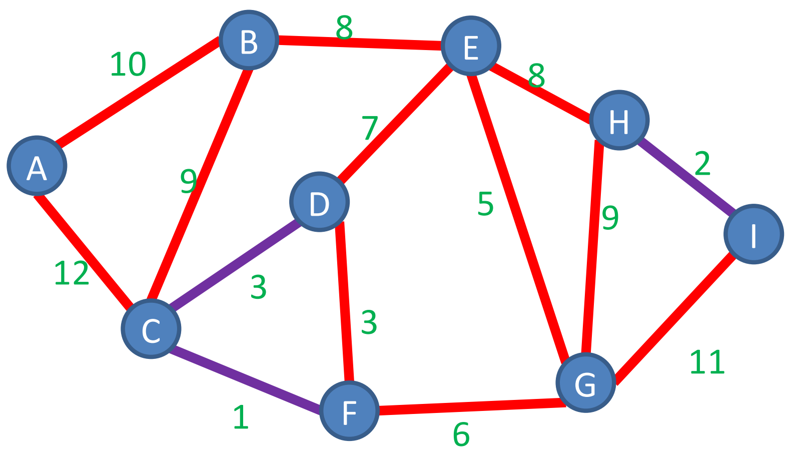
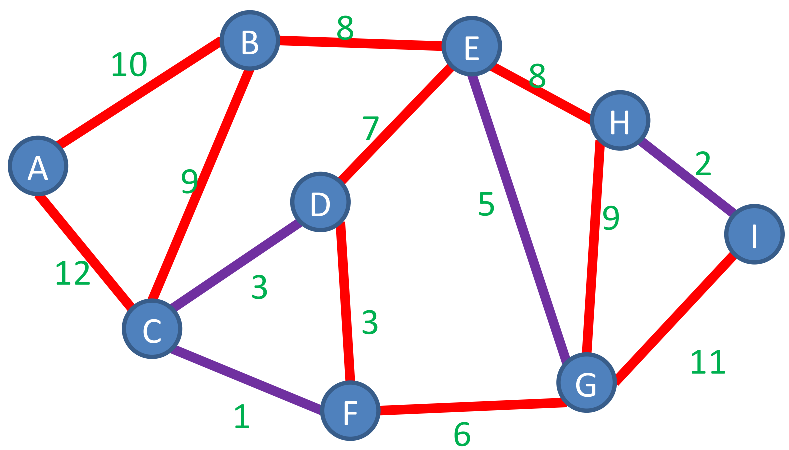


Kruskal's Algorithm

Start with an empty forest A

Add to A the lowest-weight edge that does not create a cycle



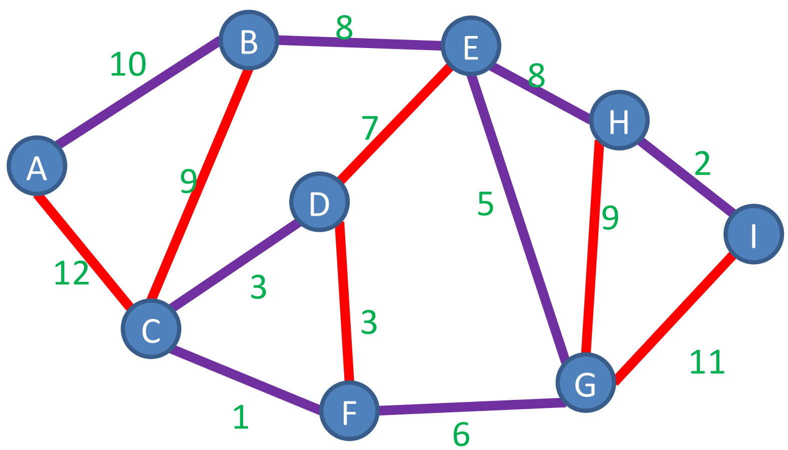
Each edge added either "grows" a tree or combines two.

Can you complete this process until we have just one tree?

Kruskal's Algorithm

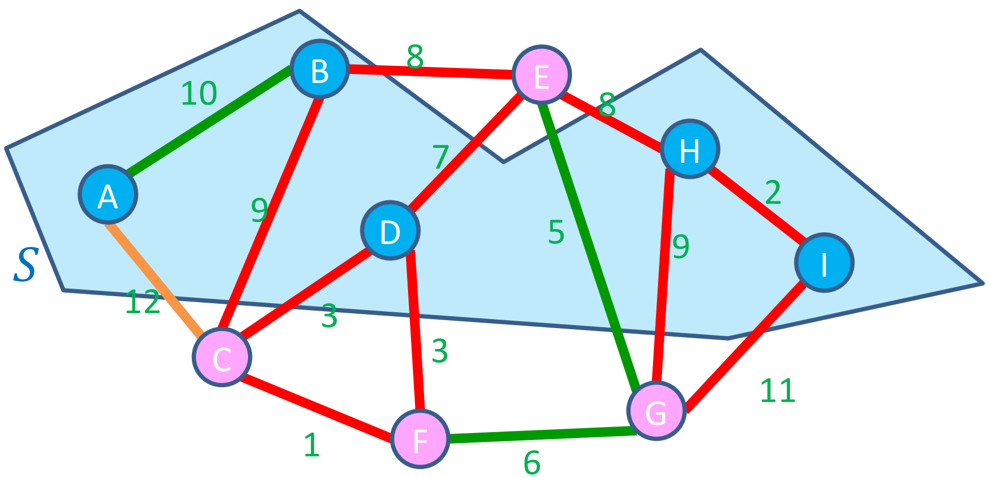
Start with an empty forest A

Add to A the lowest-weight edge that does not create a cycle



Definition: Cut

A Cut of graph $G = (V, E)$ is a partition of the nodes into two sets, S and $V - S$



Edge $(v_1, v_2) \in E$ crosses a cut if $v_1 \in S$ and $v_2 \in V - S$ (or opposite), e.g. (A, C)

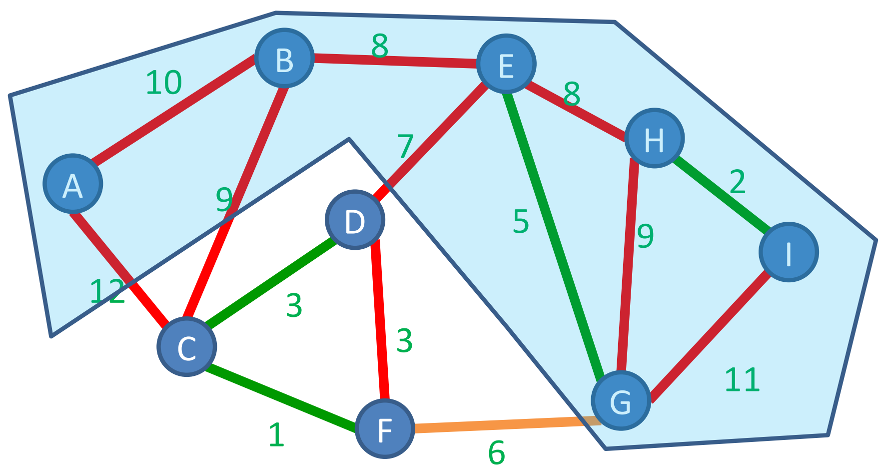
A set of edges R Respects a cut if no edges cross the cut e.g. $R = \{(A, B), (E, G), (F, G)\}$

Exchange argument

- Shows correctness of a greedy algorithm
- Idea:
  - Show exchanging an item from an arbitrary optimal solution with your greedy choice makes the new solution no worse
  - How to show my sandwich is at least as good as yours:
    - Show: “I can remove any item from your sandwich, and it would be no worse by replacing it with the same item from my sandwich”

Cut Theorem

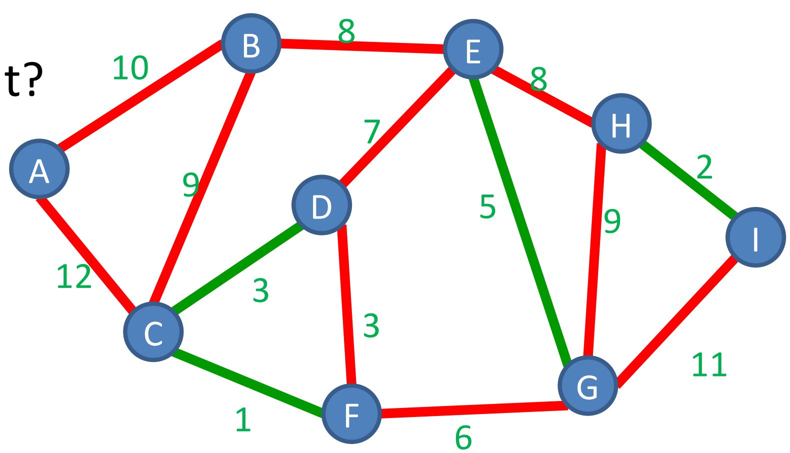
If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.



Cut Theorem : Another Example

If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

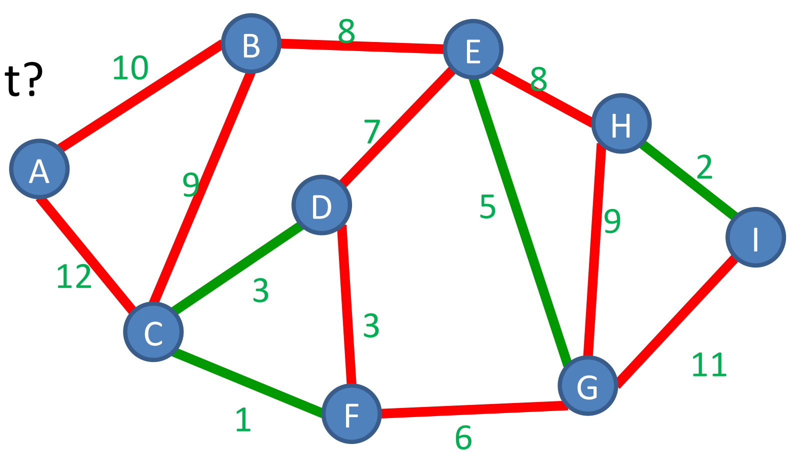
Can we draw a different cut?



Cut Theorem : Yet Another Example

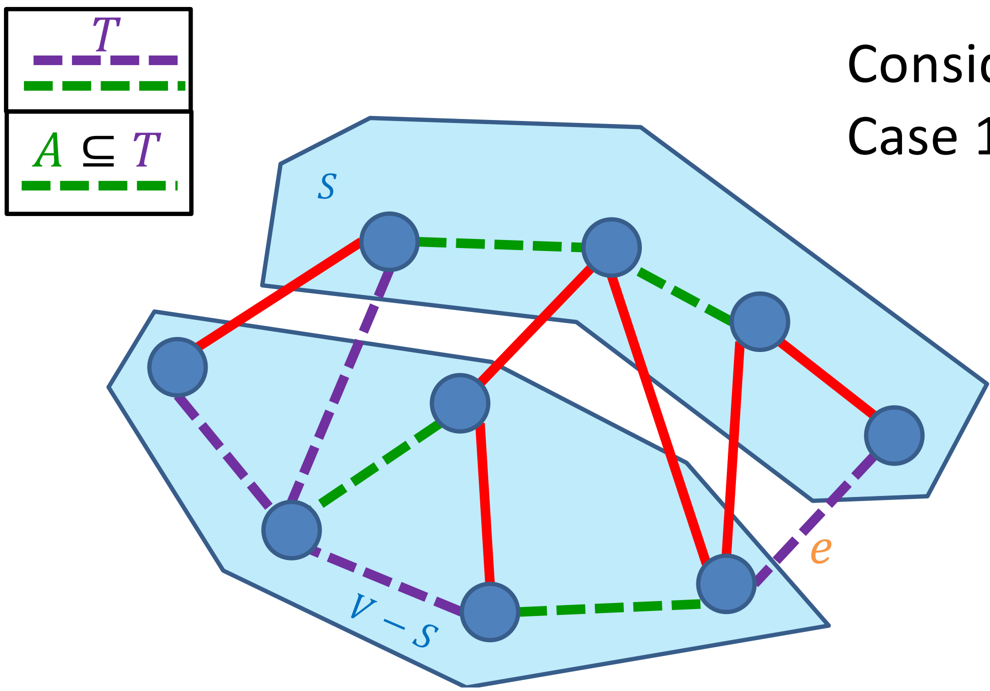
If a set of edges A is a subset of a minimum spanning tree T, let $(S, V - S)$ be any cut which A respects. Let e be the least-weight edge which crosses $(S, V - S)$. $A \cup \{e\}$ is also a subset of a minimum spanning tree.

Can we draw a different cut?



Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



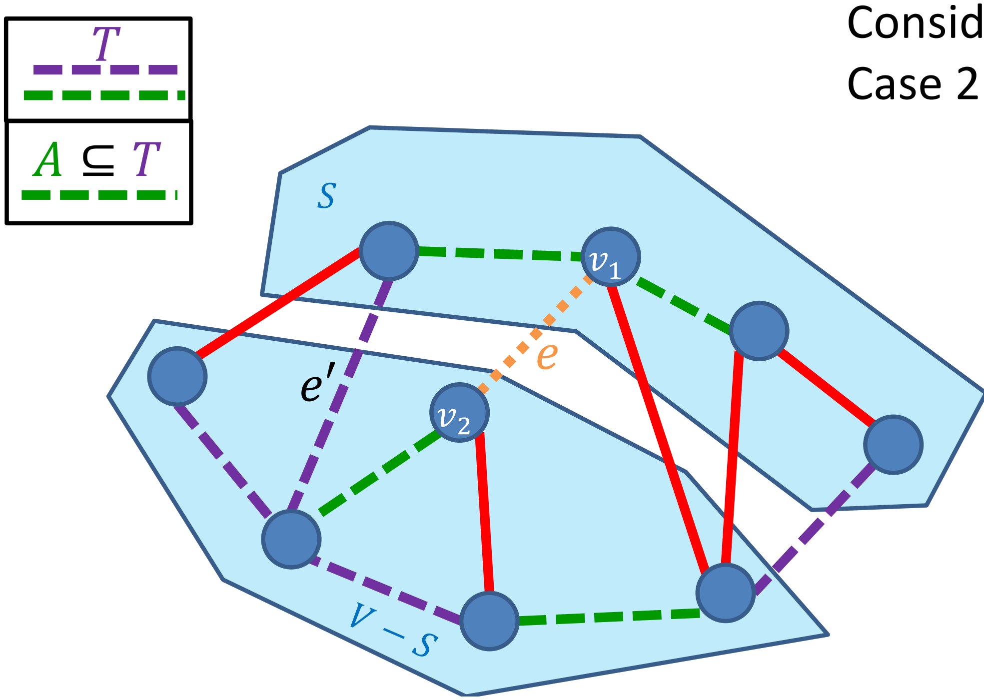
Consider some MST T,

Case 1: (the easy case)

If $e \in T$ Then claim holds

Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V-S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,

Case 2:

Consider if $e = (v_1, v_2) \notin T$

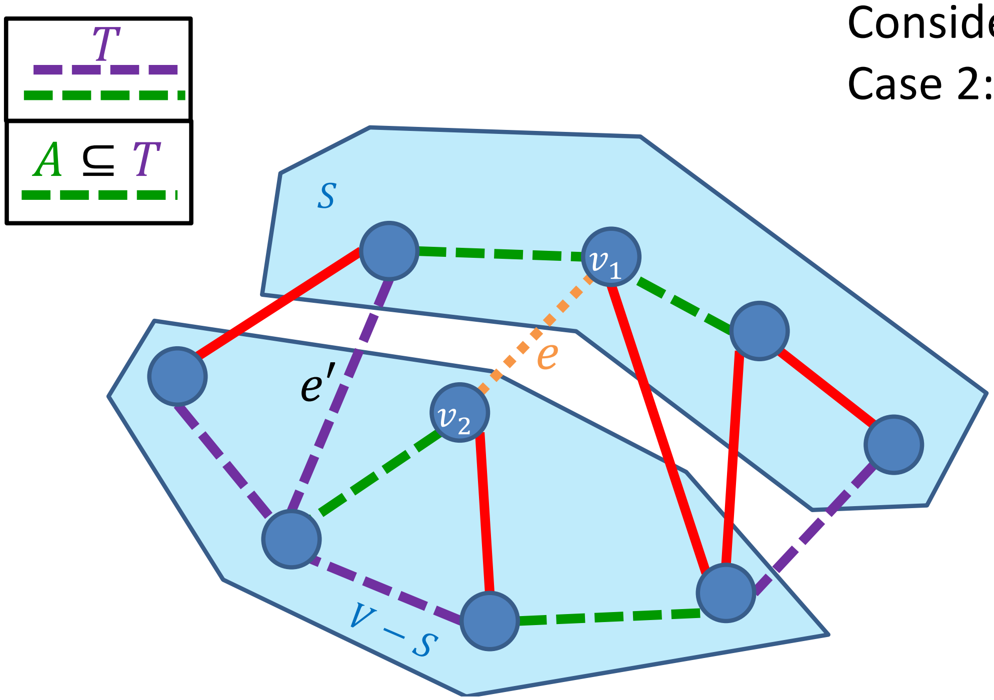
Since T is a MST, there is some path from v_1 to v_2.

Let e' be the first edge on this path which crosses the cut

Build tree T' by exchanging edge e for e'

Proof of Cut Theorem

Claim: If A is a subset of a MST T, and e is the least-weight edge which crosses cut $(S, V - S)$ (which A respects) then $A \cup \{e\}$ is also a subset of a MST.



Consider some MST T,

Case 2:

Consider if $e = (v_1, v_2) \notin T$

$T' = T$ with edge e instead of e'

We assumed $w(e) \leq w(e')$

$w(T') = w(T) - w(e') + w(e)$

$w(T') \leq w(T)$

So T' is also a MST!

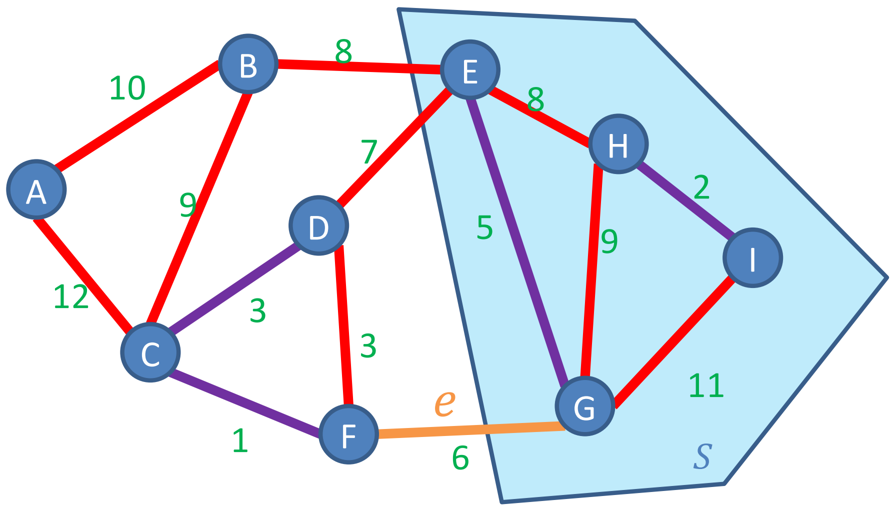
Thus the claim holds

Kruskal's Algorithm: Time Complexity?

Start with an empty forest A

Repeat $V - 1$ times:

Add the min-weight edge that doesn't cause a cycle



First, need to sort edges.

At each step:

- Does edge connect nodes in same tree?
- If not, "union" nodes in two trees to make one.

Problem: Union/Find for sets

Solution: Keep edges in a Disjoint-set data structure (very fancy)

$$O(E \log V)$$

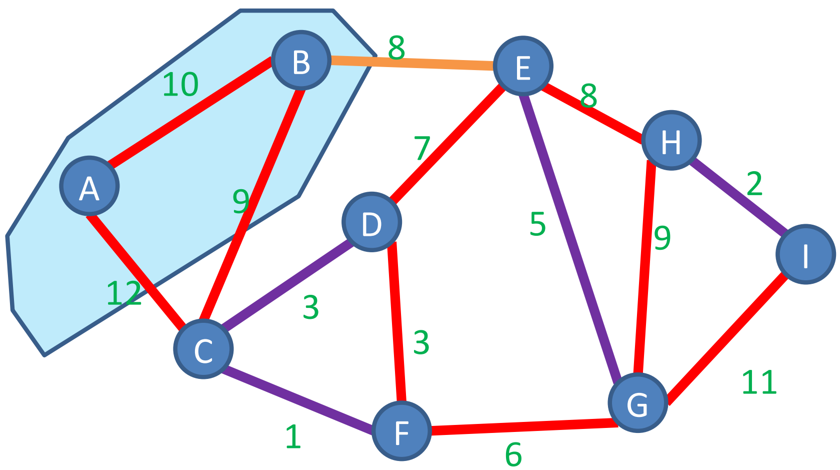
General MST Algorithm

Start with an empty tree A

Repeat $V - 1$ times:

  Pick a cut $(S, V - S)$ which A respects

  Add the min-weight edge which crosses $(S, V - S)$



Prim's Algorithm

Start with an empty tree A

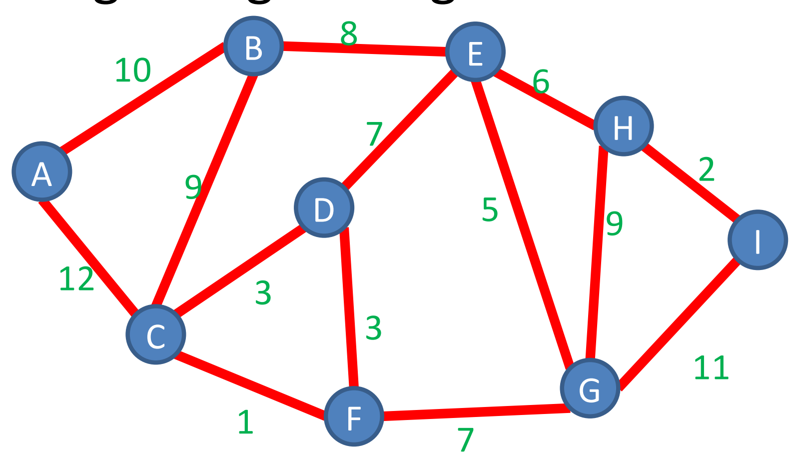
Repeat $V - 1$ times:

    Pick a cut $(S, V - S)$ which A respects

    Add the min-weight edge which crosses $(S, V - S)$

S is set of nodes that are endpoints of edges in A

e is the min-weight edge that grows the tree



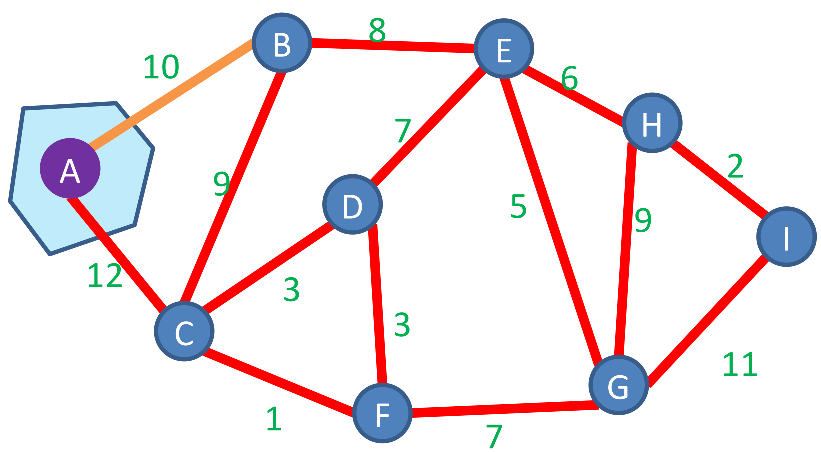
Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A



Prim's Algorithm

Start with an empty tree A

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Repeat $V - 1$ times:

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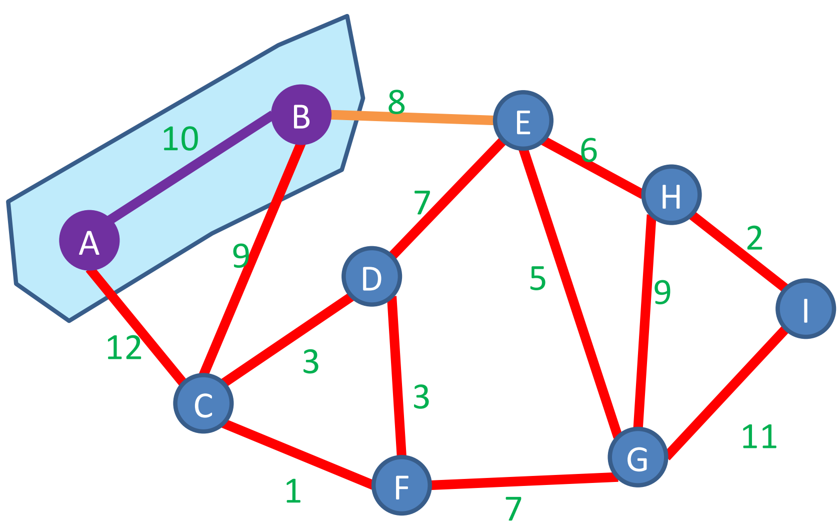

Prim's Algorithm

Start with an empty tree A

Pick a start node

Repeat $V - 1$ times:

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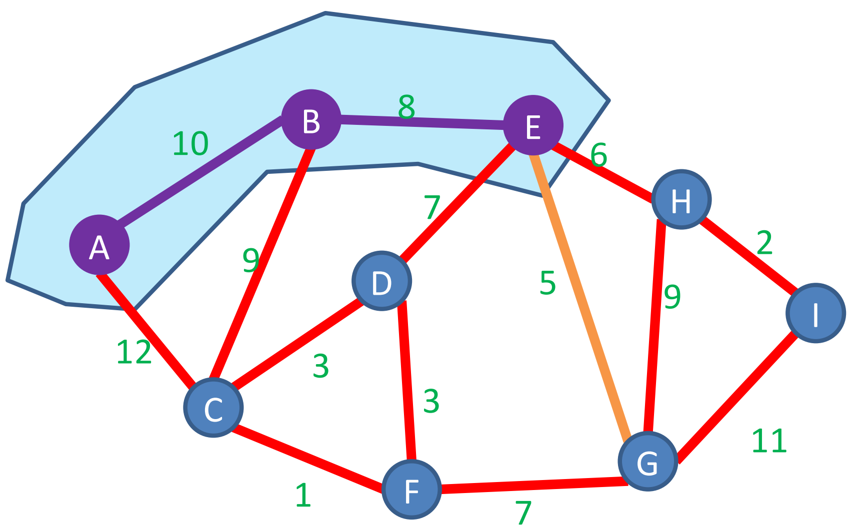

Prim's Algorithm

Start with an empty tree A

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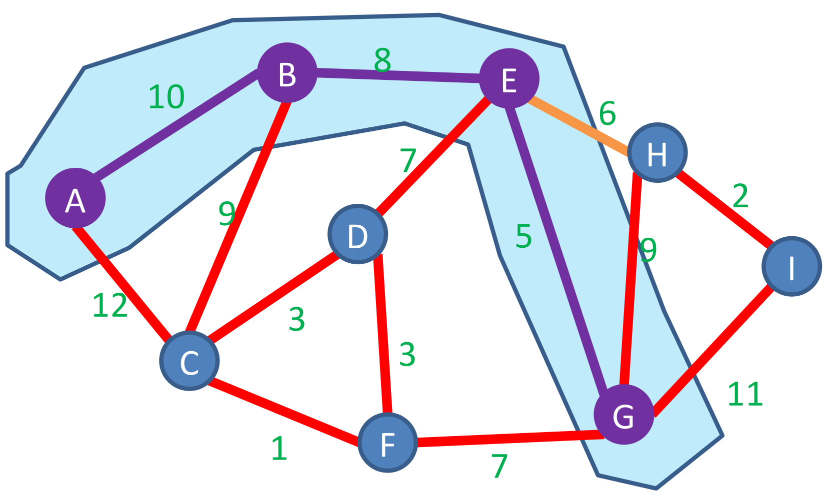


Prim's Algorithm

Start with an empty tree A

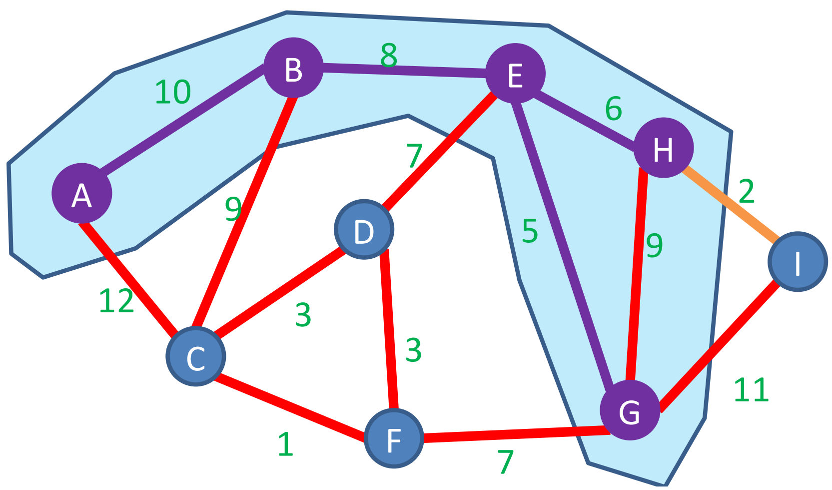
Pick a start node

Repeat $V - 1$ times:

Add the min-weight edge which connects to node in A with a node not in A

Keep edges in a Heap

$O(E \log V)$



Can you finish this?

Summary of MST results

- Fredman-Tarjan ‘84: $\Theta(E + V \log V)$
- Gabow et al ‘86: $\Theta(E \log \log^* V)$
- Chazelle ‘00: $\Theta(E\alpha(V))$
- Pettie-Ramachandran ’02: $\Theta(?)$(optimal)
- Karger-Klein-Tarjan ‘95: $\Theta(E)$ (randomized)

- [read and summarize any/all for EC]
- [read and summarize about union/find for sets for EC]