CS4102 Algorithms

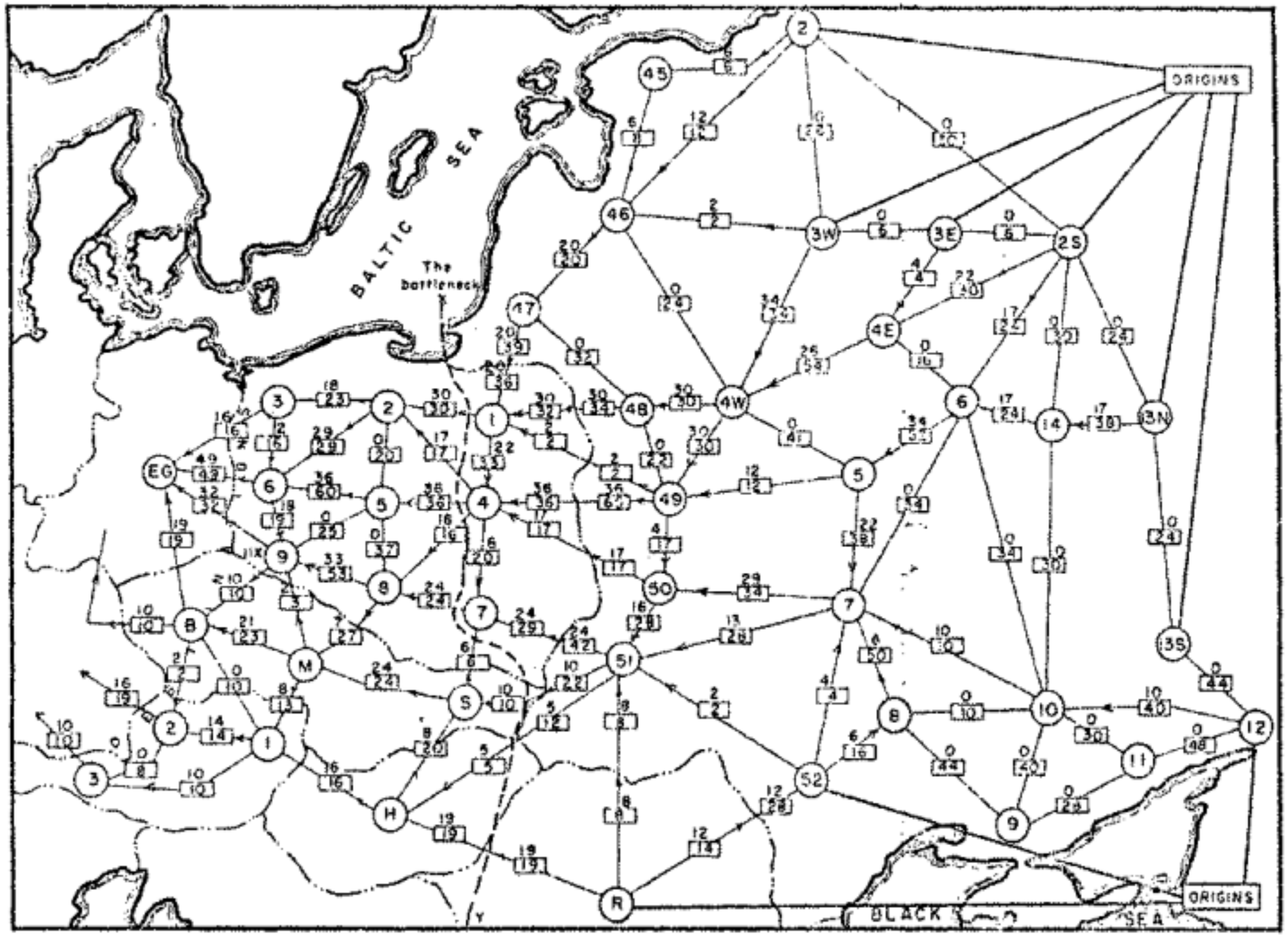
Spring 2020

Today's Keywords

- Graphs
- **MaxFlow/MinCut**
- Ford-Fulkerson
- Edmunds-Karp

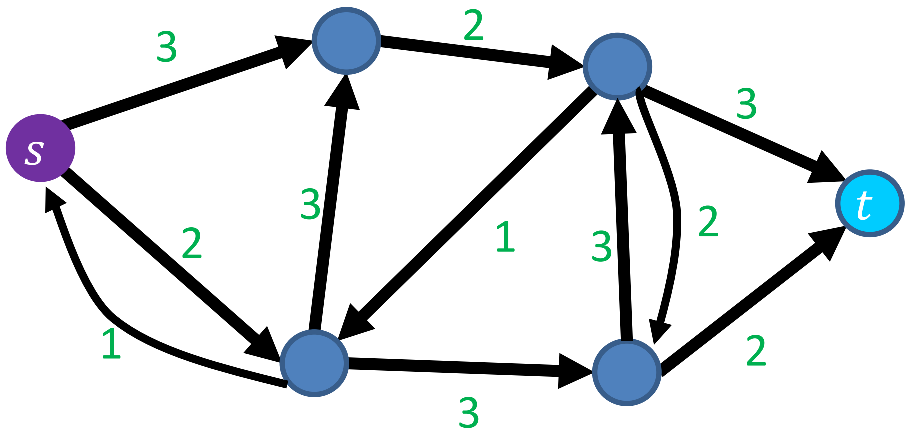
CLRS Readings

- Chapter 25, 26



Railway map of Western USSR, 1955

Flow Network



Graph $G = (V, E)$

Source node $s \in V$

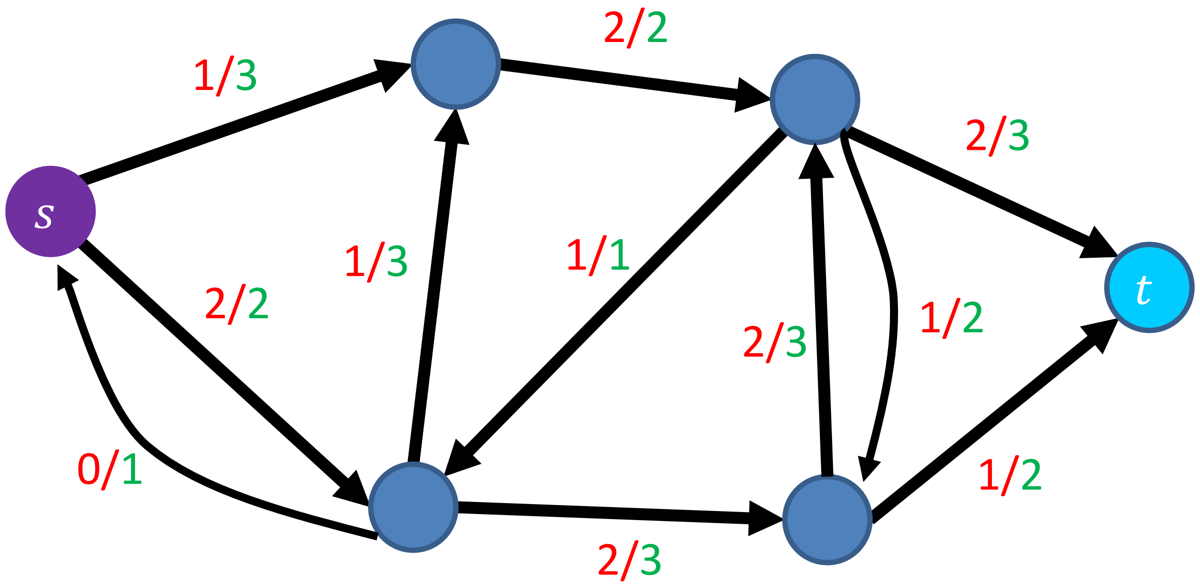
Sink node $t \in V$

Edge Capacities $c(e) \in$ Positive Real numbers

Max flow intuition: If s is a faucet, t is a drain, and s connects to t through a network of pipes with given capacities, what is the maximum amount of water which can flow from the faucet to the drain?

Flow

- Assignment of values to edges
  - $f(e) = n$
  - Amount of water going through that pipe
- Capacity constraint
  - $f(e) \le c(e)$
  - Flow cannot exceed capacity
- Flow constraint
  - $\forall v \in V - \{s,t\}, inflow(v) = outflow(v)$
  - $inflow(v) = \sum_{x \in V} f(v,x)$
  - $outflow(v) = \sum_{x \in V} f(x,v)$
  - Water going in must match water coming out
- Flow of G: $|f| = outflow(s) - inflow(s)$
  - Net outflow of s



Flow/Capacity

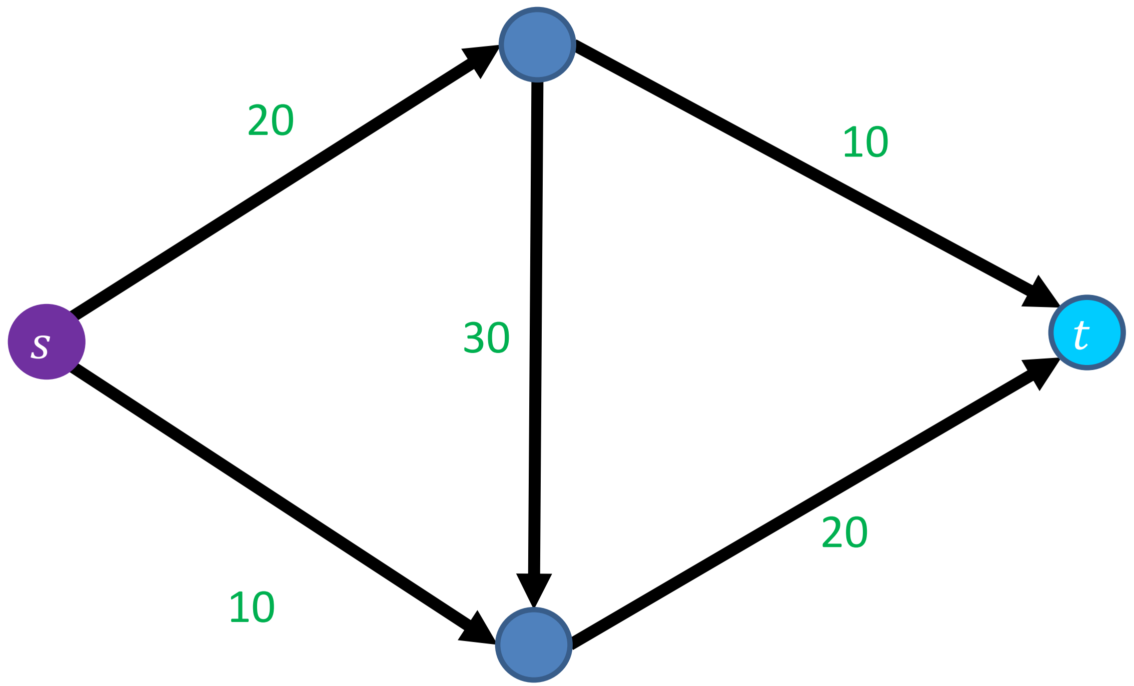
3 in example above

Max Flow

- Of all valid flows through the graph, find the one which maximizes:
  - $|f| = outflow(s) - inflow(s)$

Greedy doesn't work

Saturate Highest Capacity Path First



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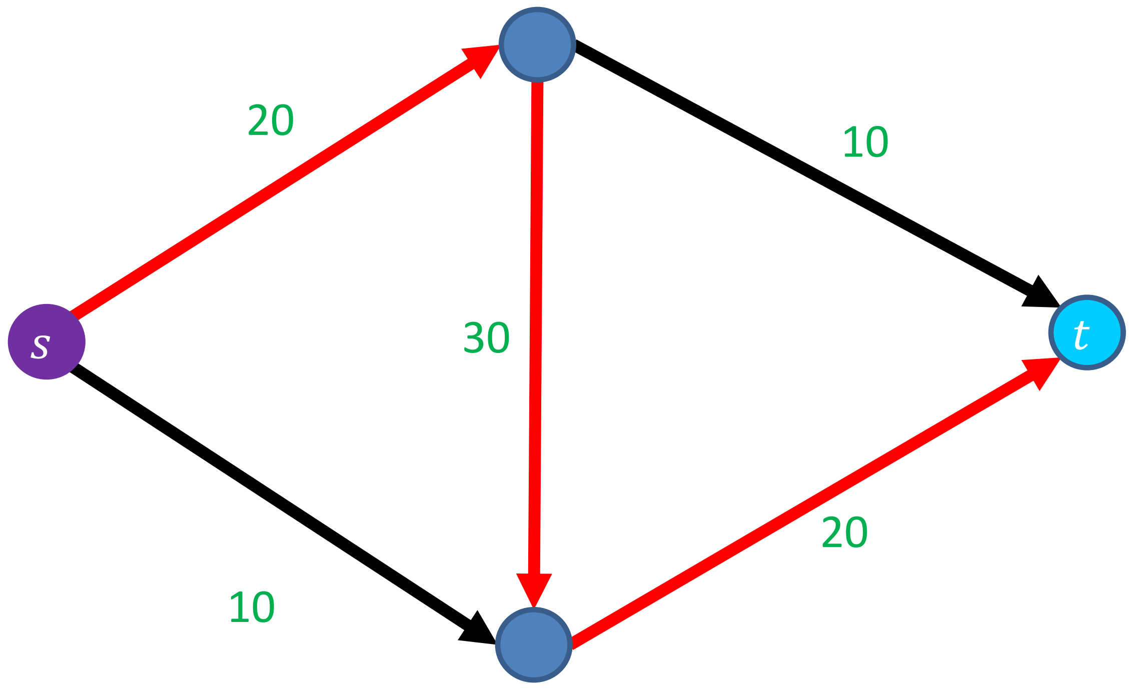

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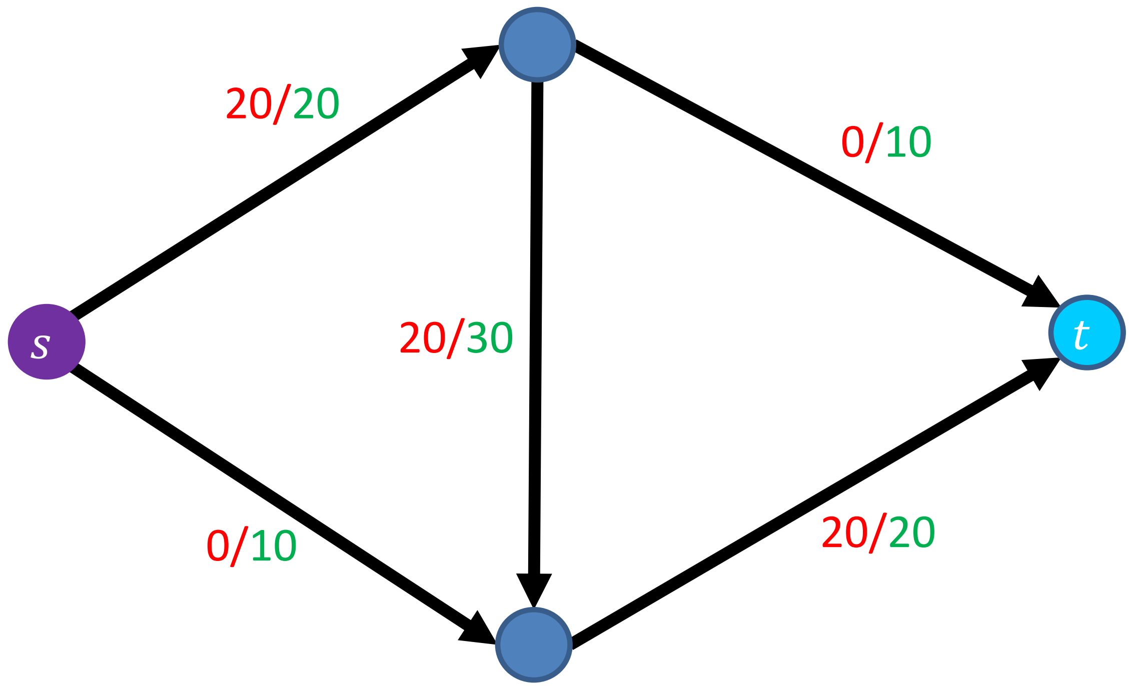
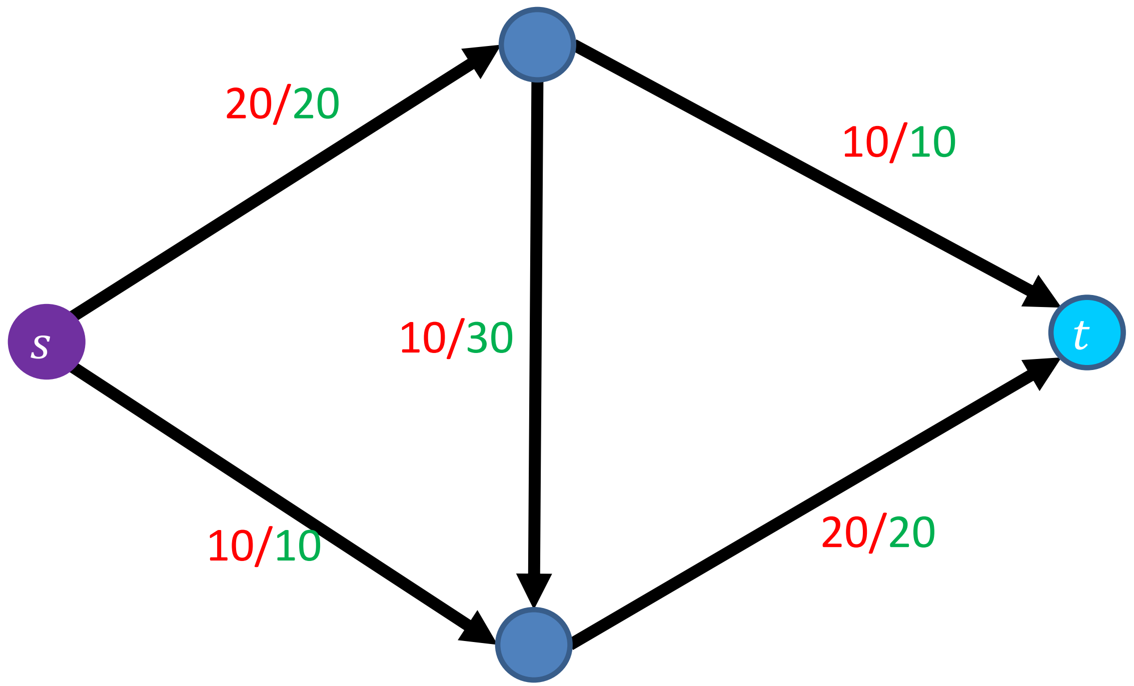


Overall Flow: $|f| = 20$

Greedy doesn't work

Better Solution



Overall Flow: $|f| = 30$

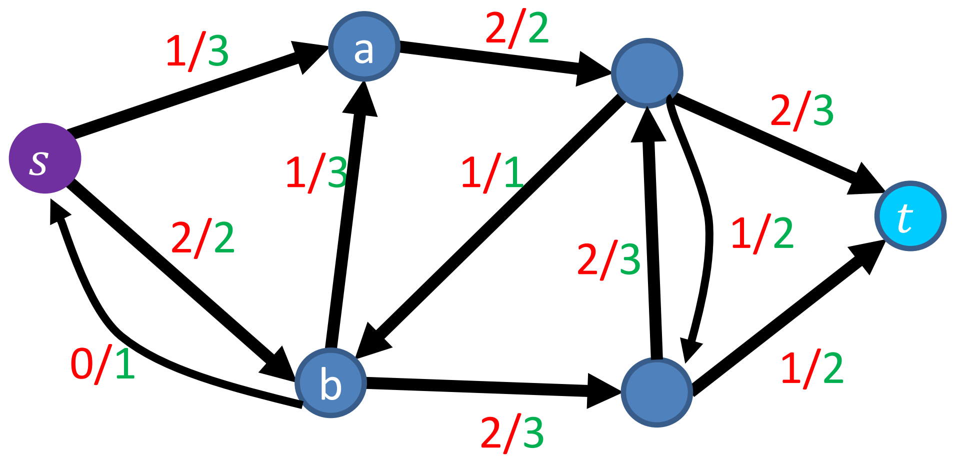
Residual Graph G_f

- Keep track of net available flow along each edge
- "Forward edges": weight is equal to available flow along that edge in the flow graph
  - $w(e) = c(e) - f(e)$
- "Back edges": weight is equal to flow along that edge in the flow graph
  - $w(e) = f(e)$

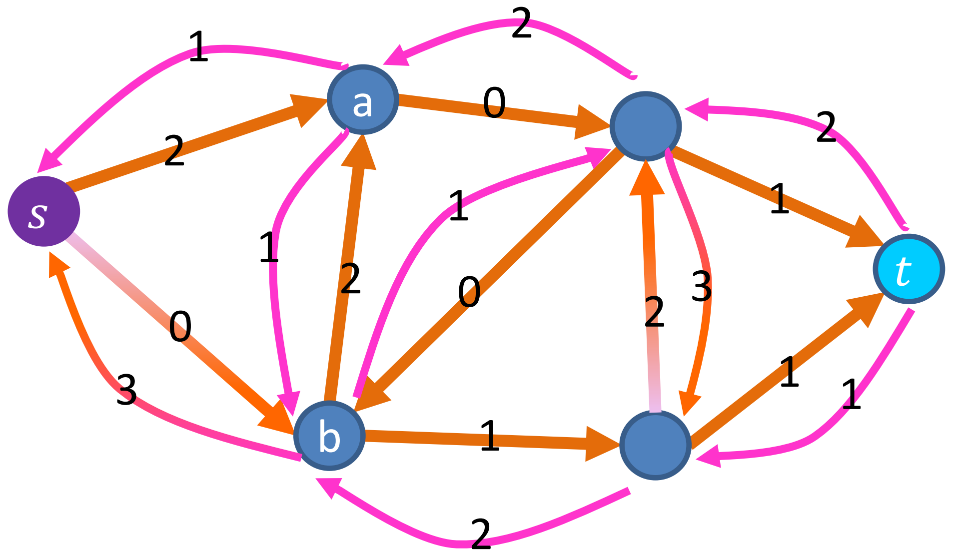
Flow I *could* add

Flow I *could* remove

Flow Graph G

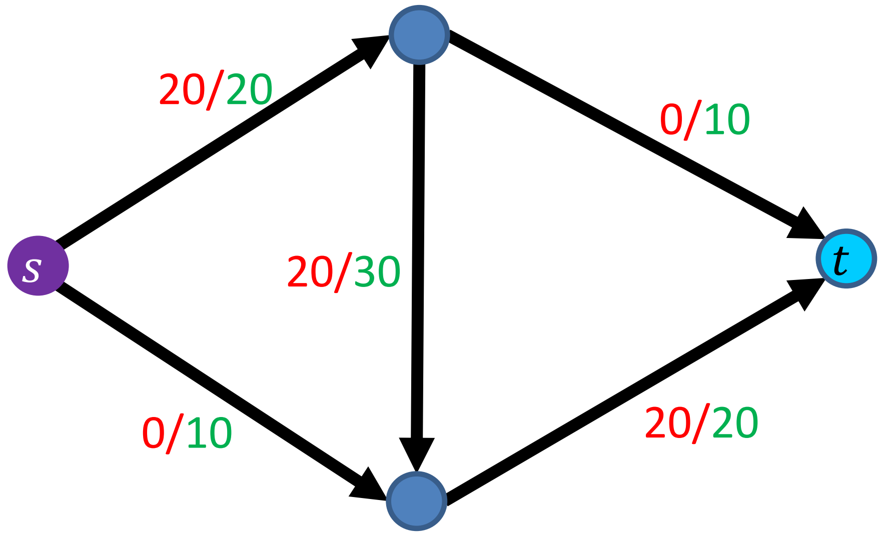


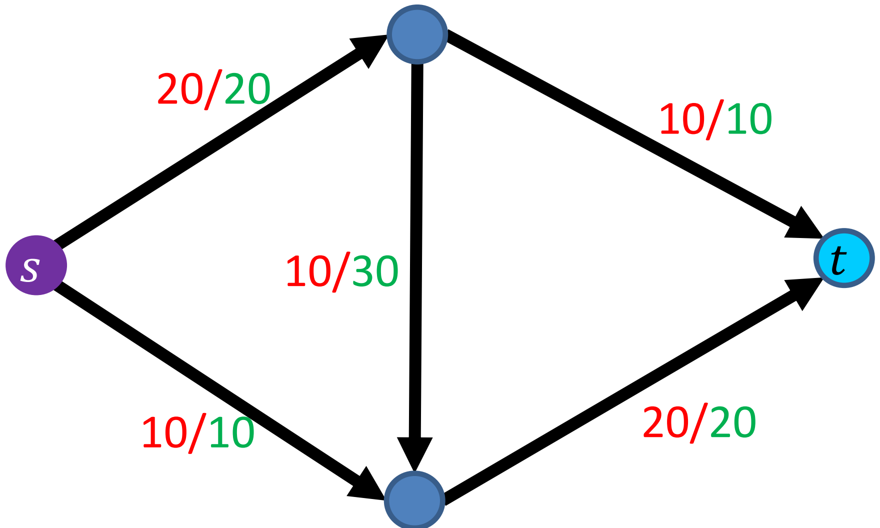
Residual Graph G_f



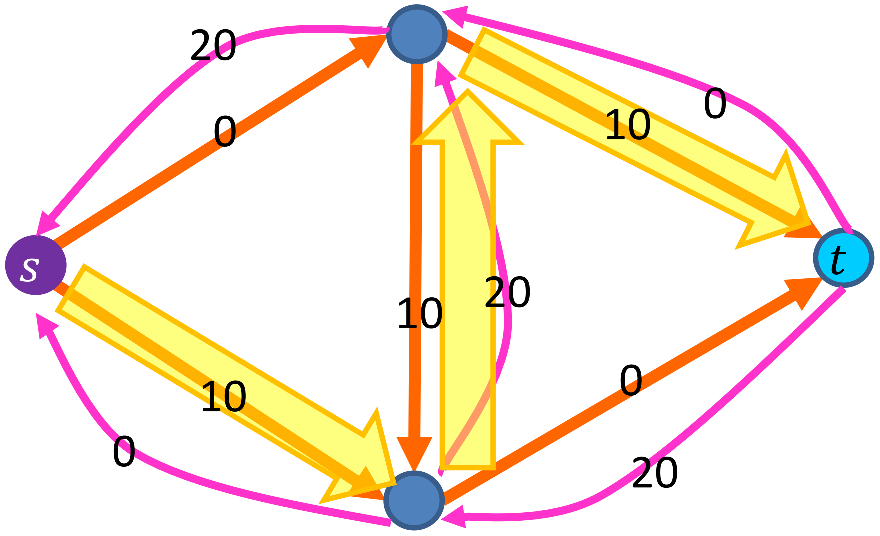
Residual Graphs Example

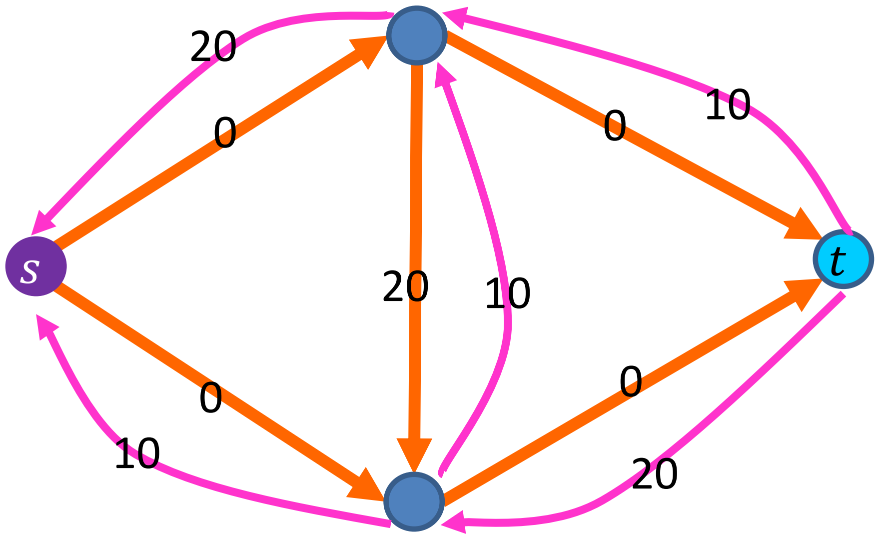
Flow Graph





Residual Graph





Ford-Fulkerson Algorithm

Define an **augmenting path** to be a path from $s \rightarrow t$ in the residual graph G_f (using edges of non-zero weight)

Overview: Repeatedly add the flow of any augmenting path

Ford-Fulkerson max-flow algorithm:

- Initialize $f(e) = 0$ for all $e \in E$
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Ford-Fulkerson approach: take any augmenting path (will revisit this later)

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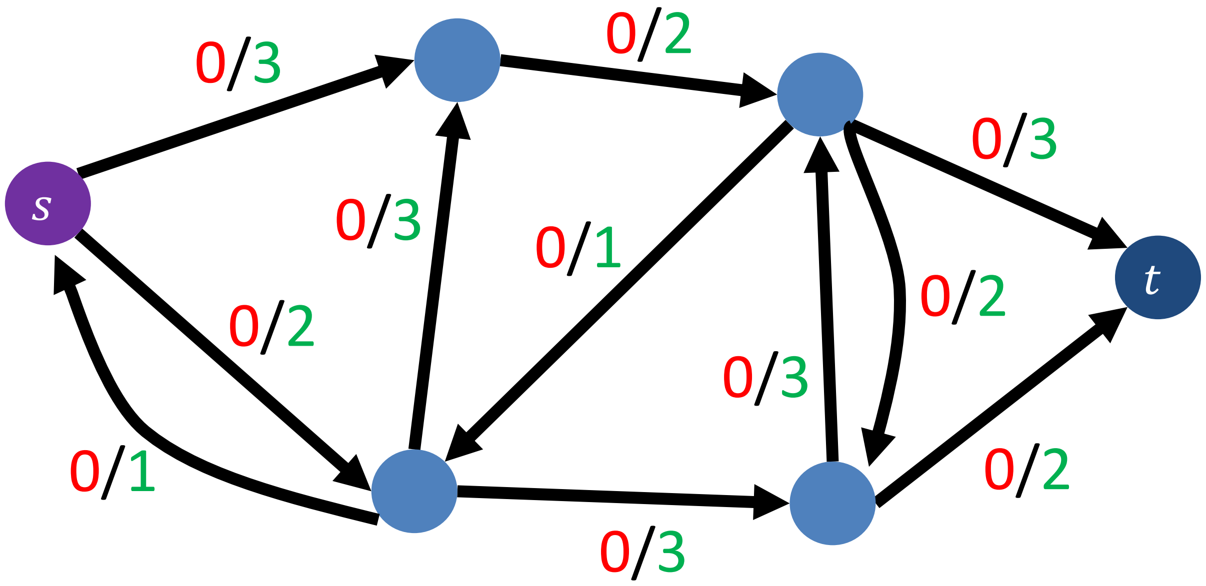
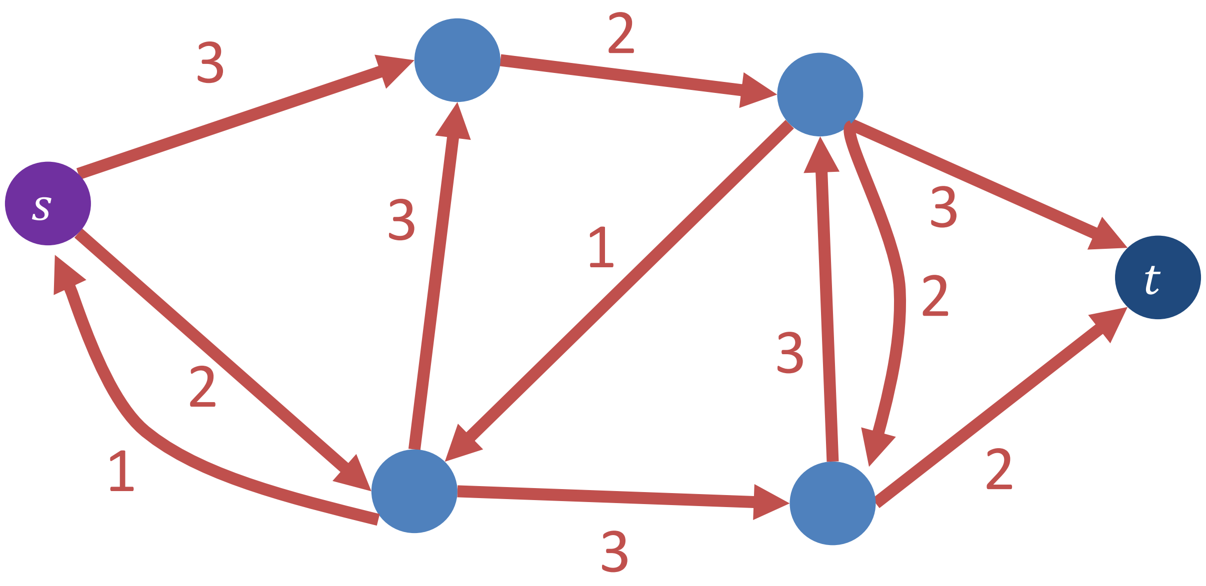
($c_f(u, v)$ is the weight of edge (u, v) in the residual network G_f)

Ford-Fulkerson Example

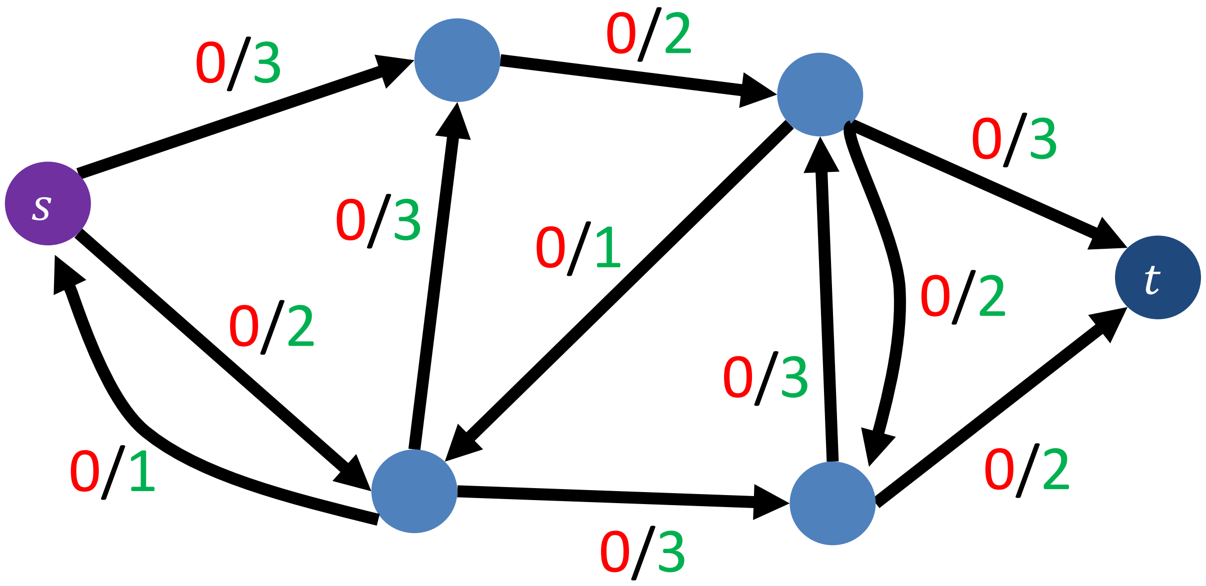




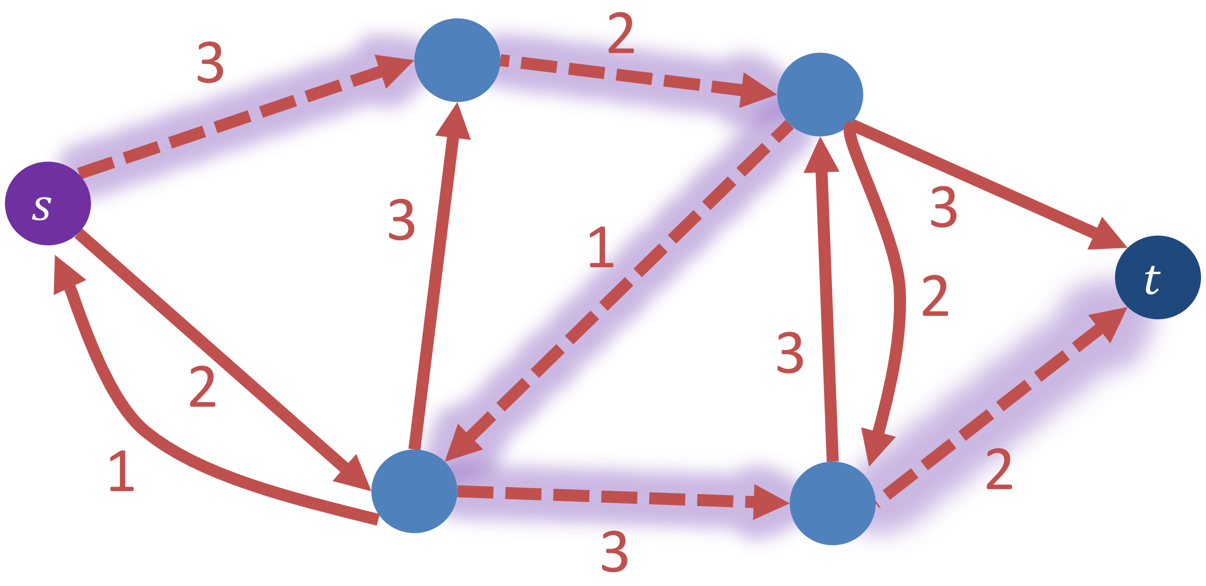
Initially: $f(e) = 0$ for all $e \in E$

Residual graph G_f

Ford-Fulkerson Example



Increase flow by 1 unit



Residual graph G_f

Ford-Fulkerson Example

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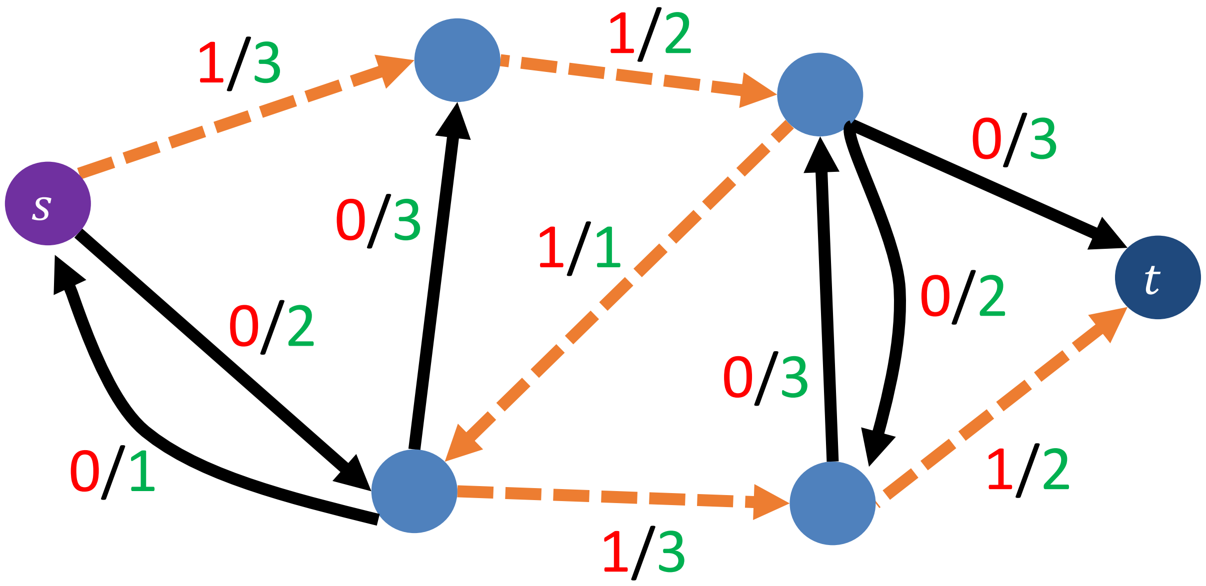
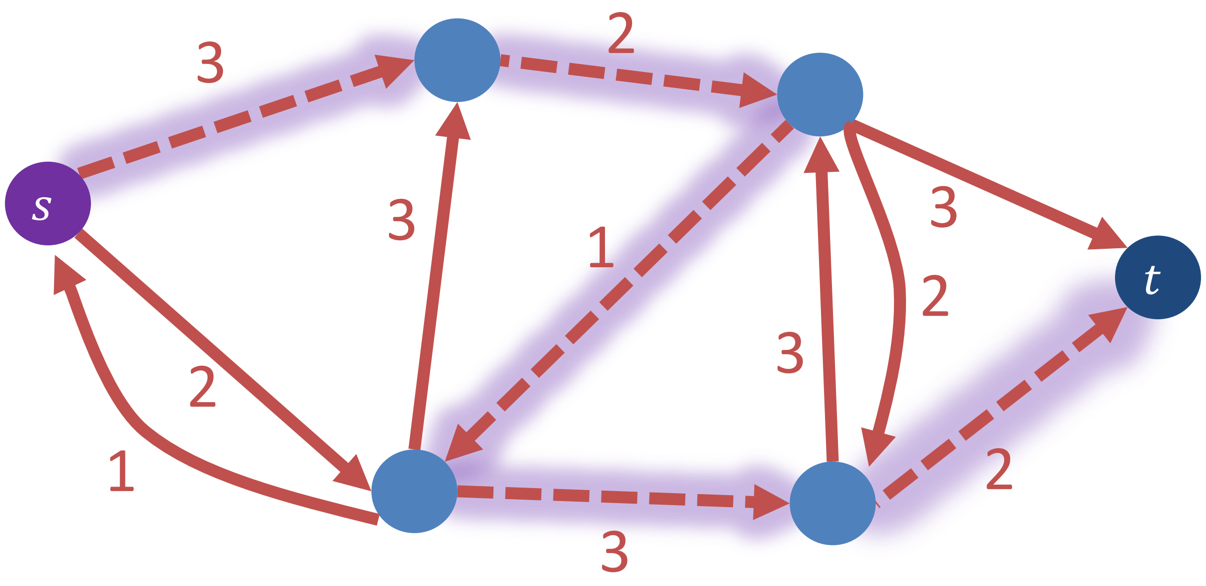
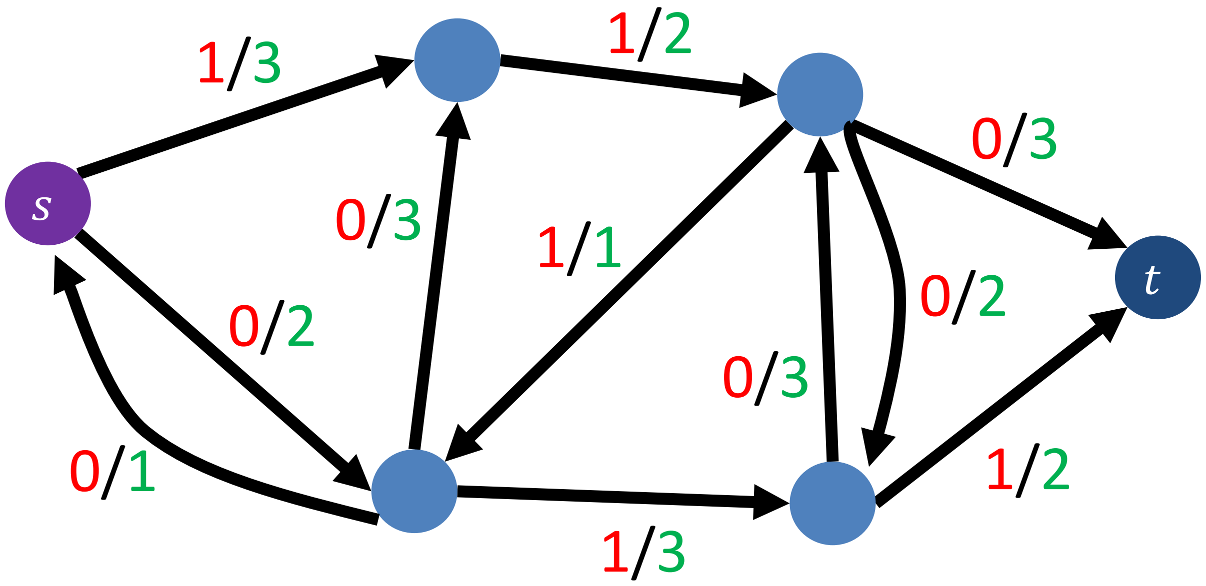
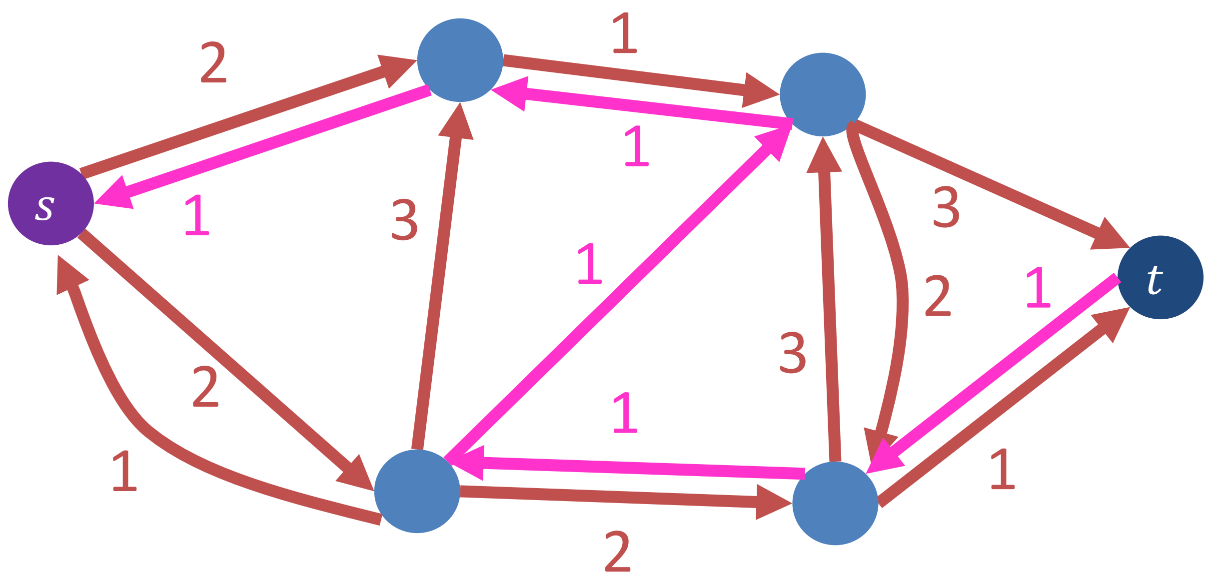




Residual graph G_f

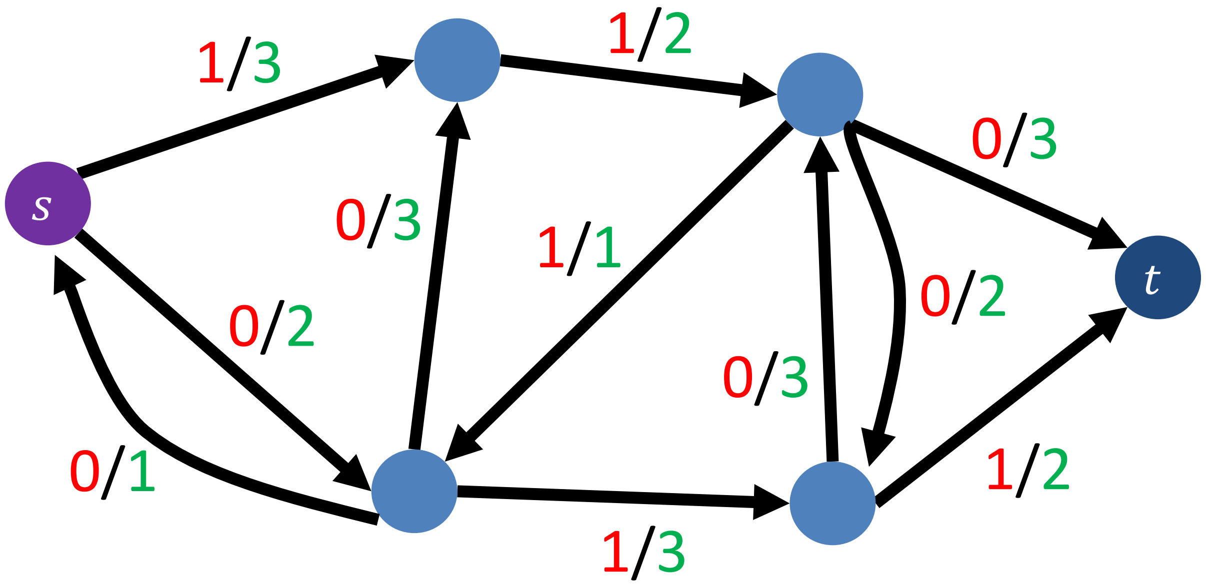
Ford-Fulkerson Example





Residual graph G_f

Ford-Fulkerson Example



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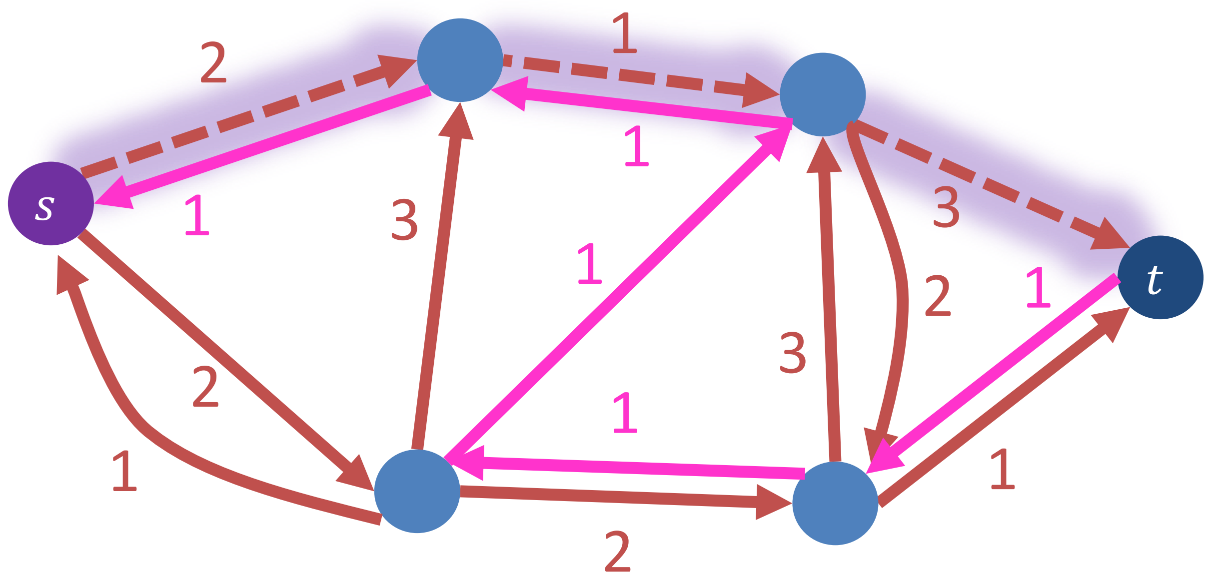

Residual graph G_f

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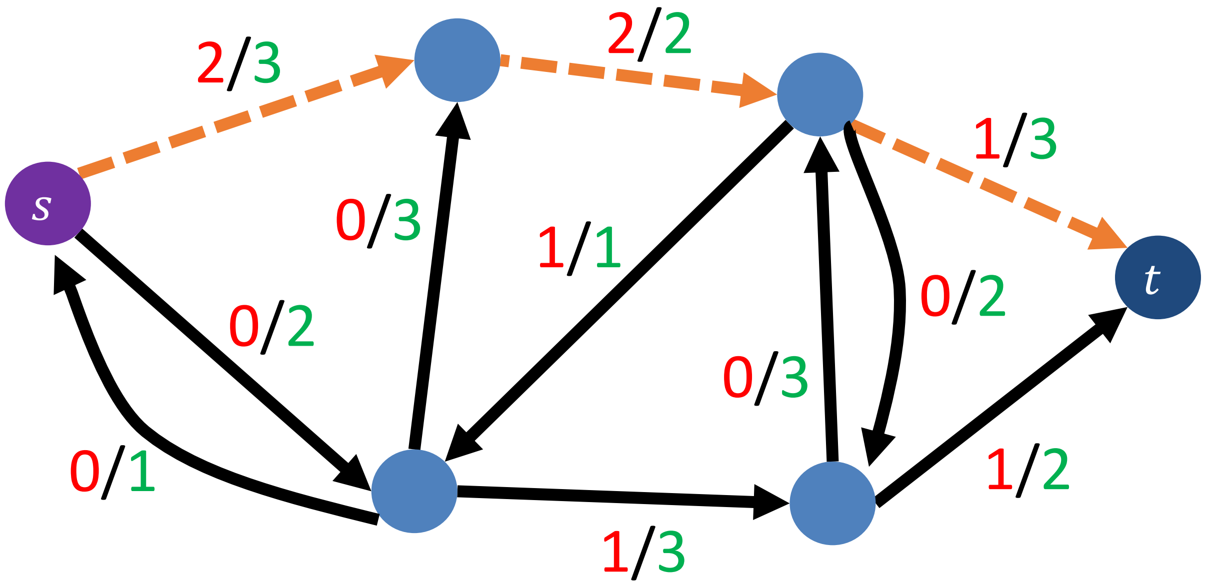

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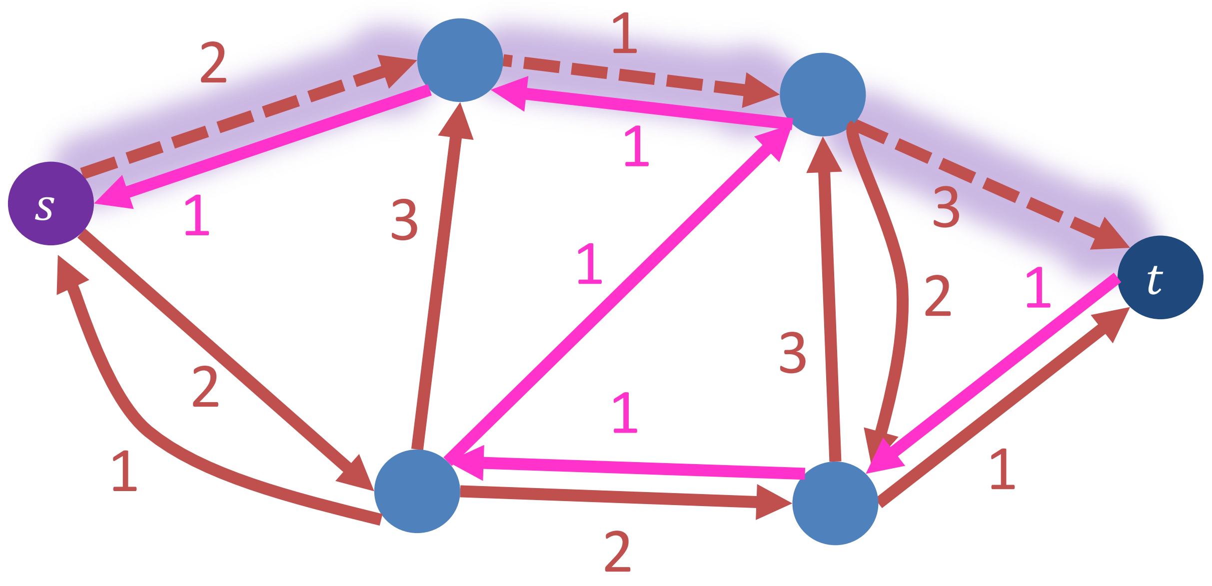

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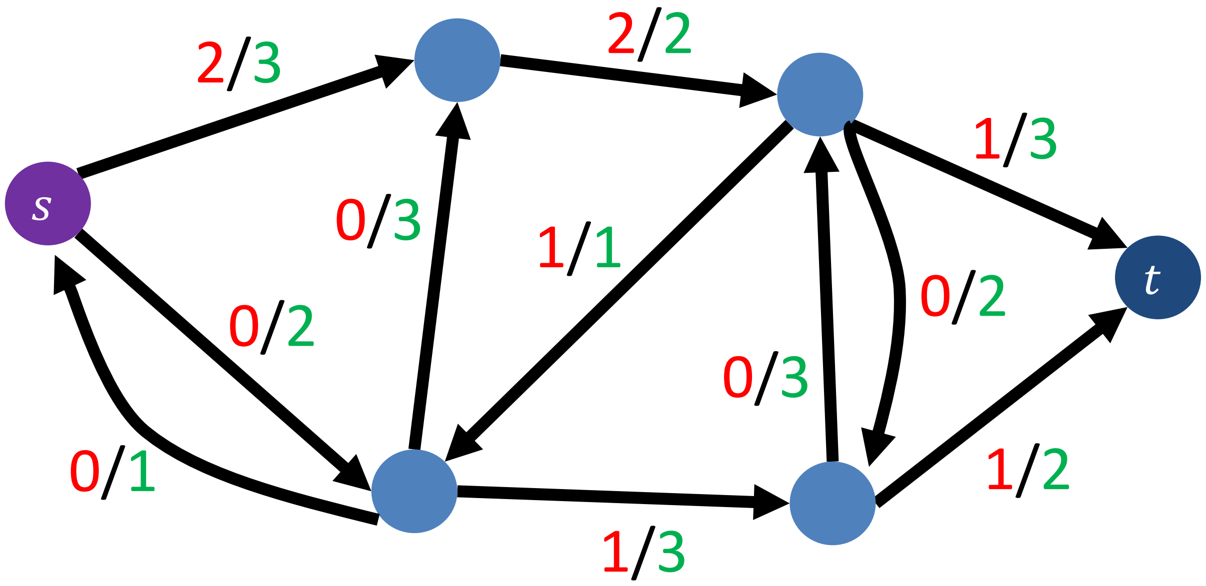

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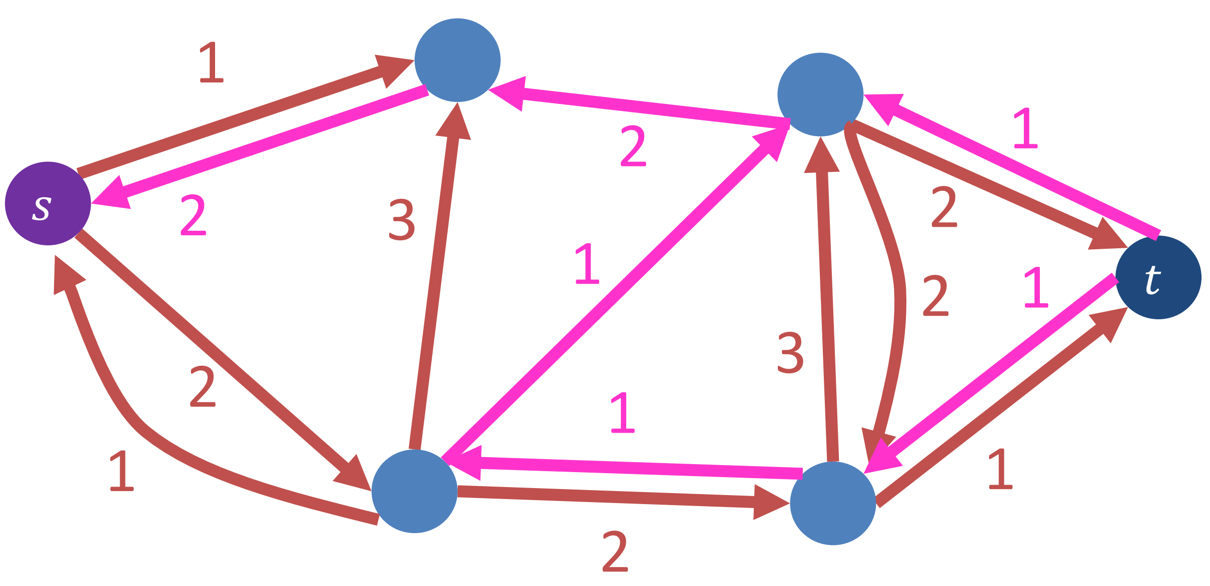

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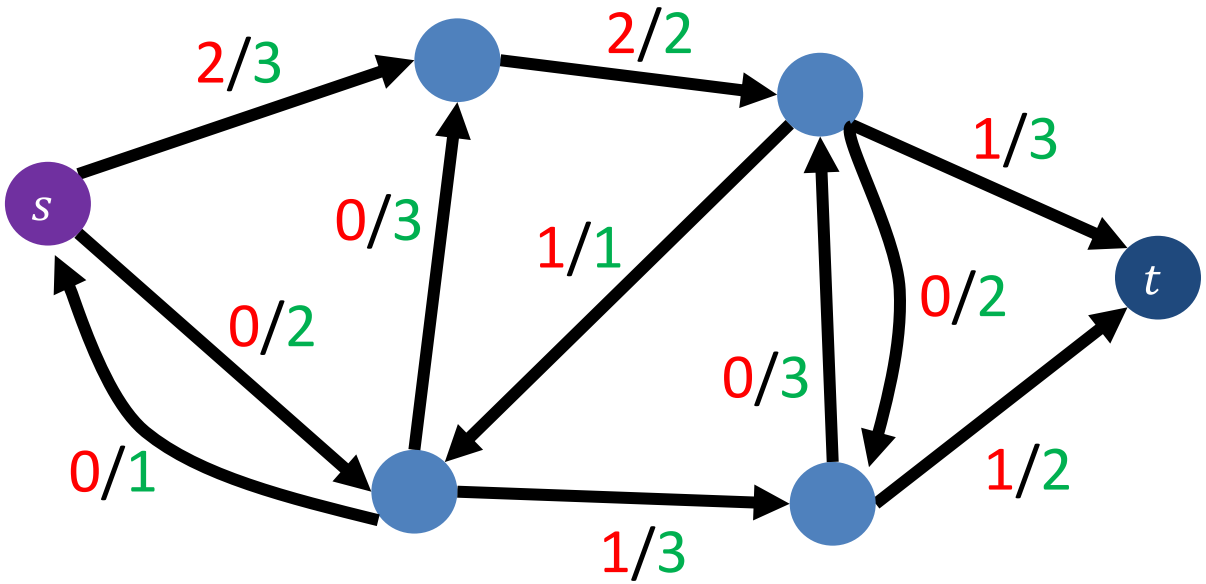

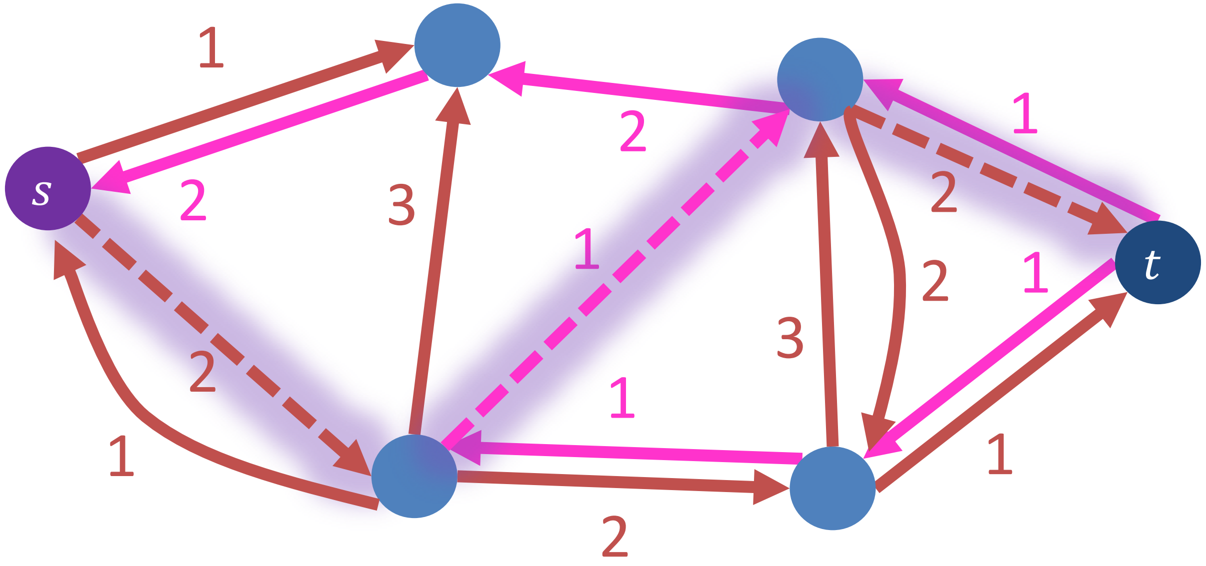

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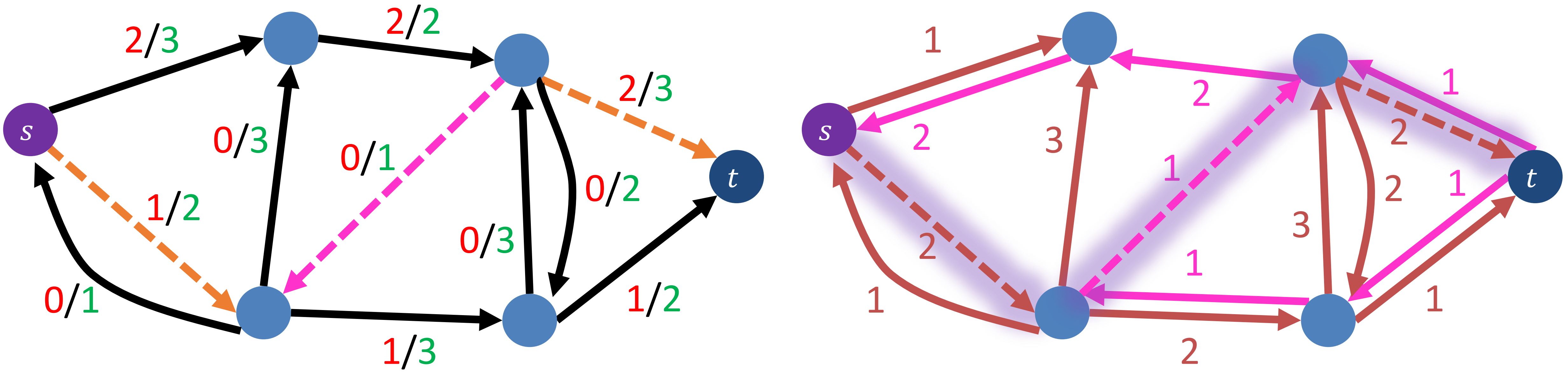

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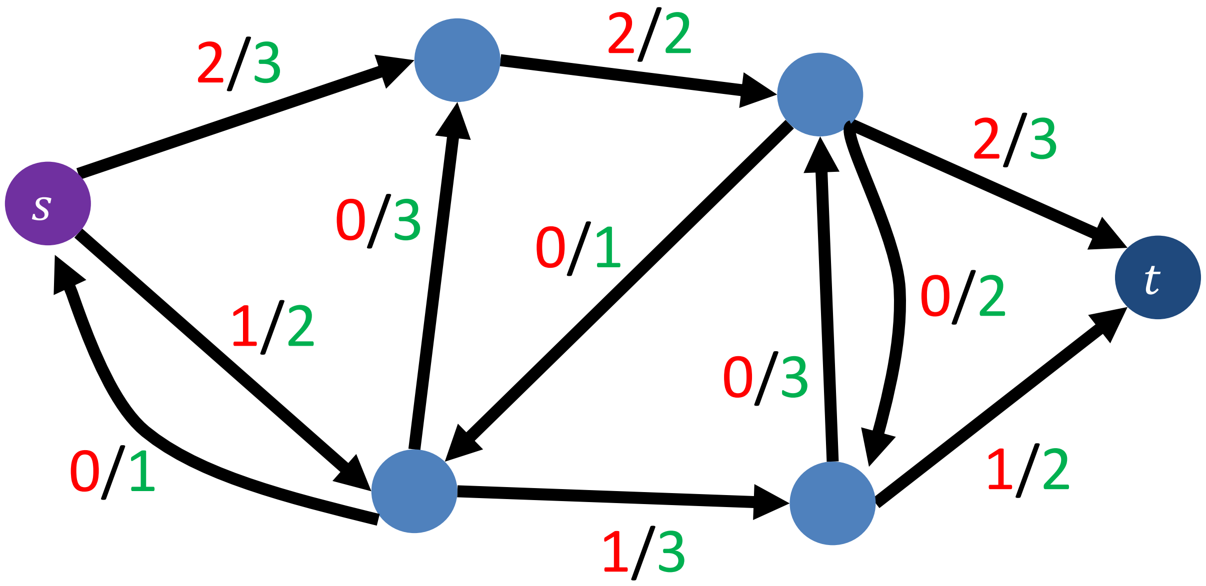

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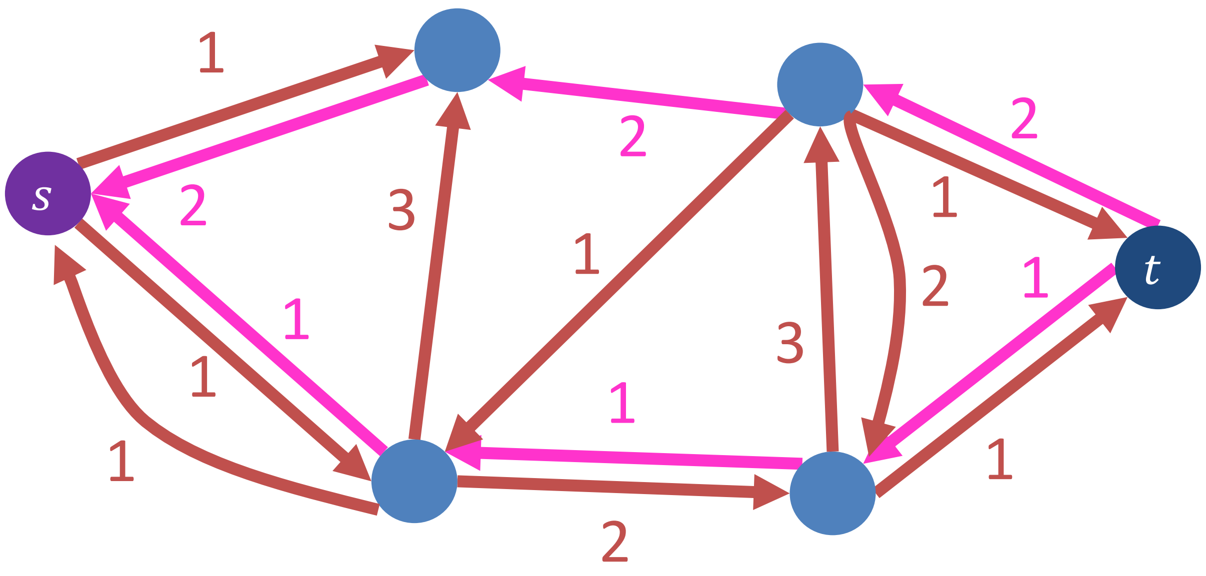

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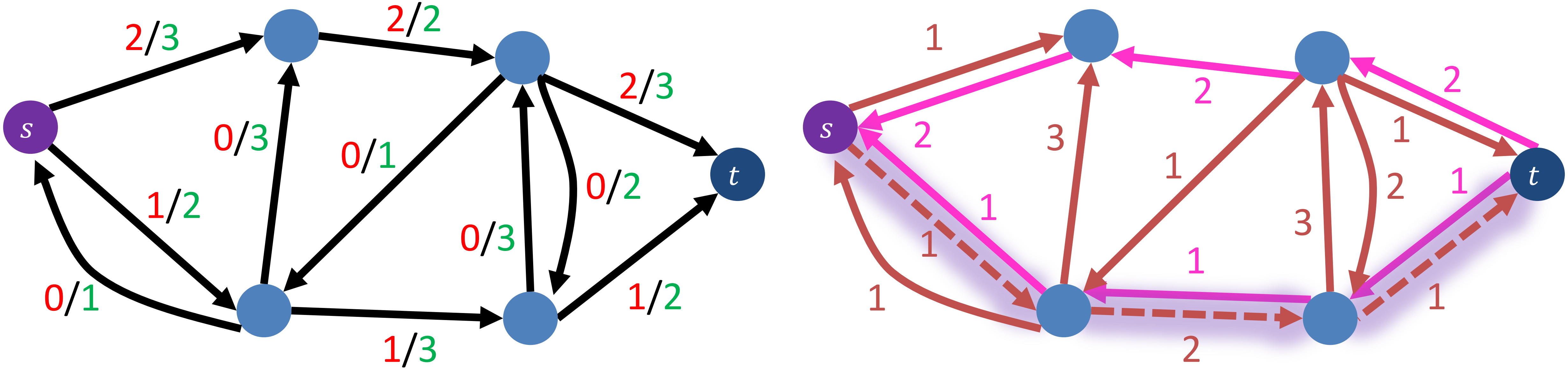

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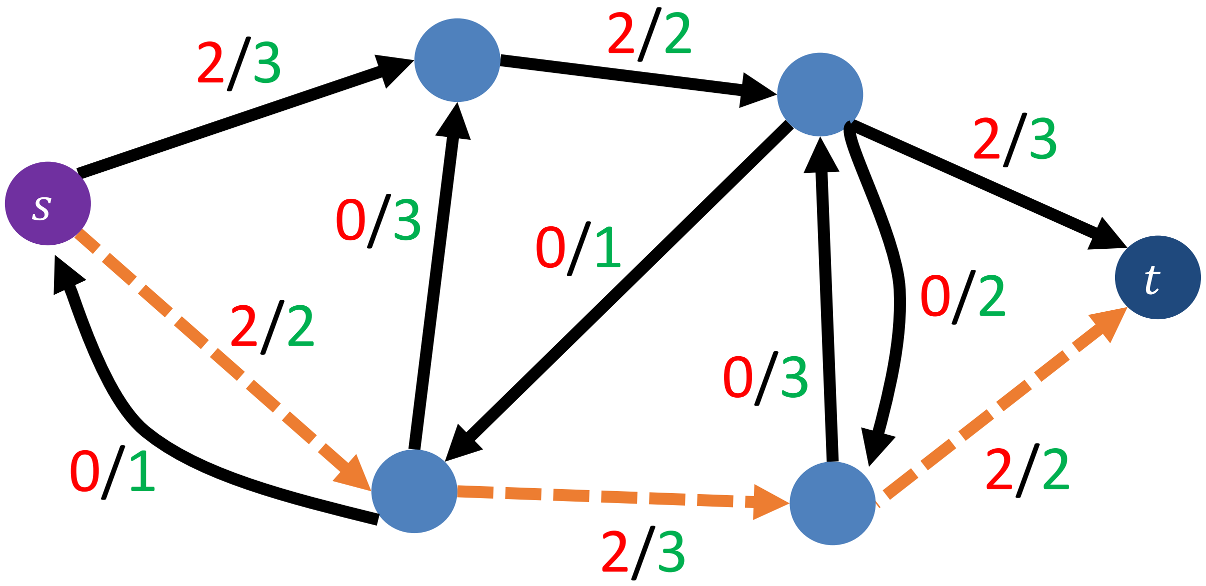

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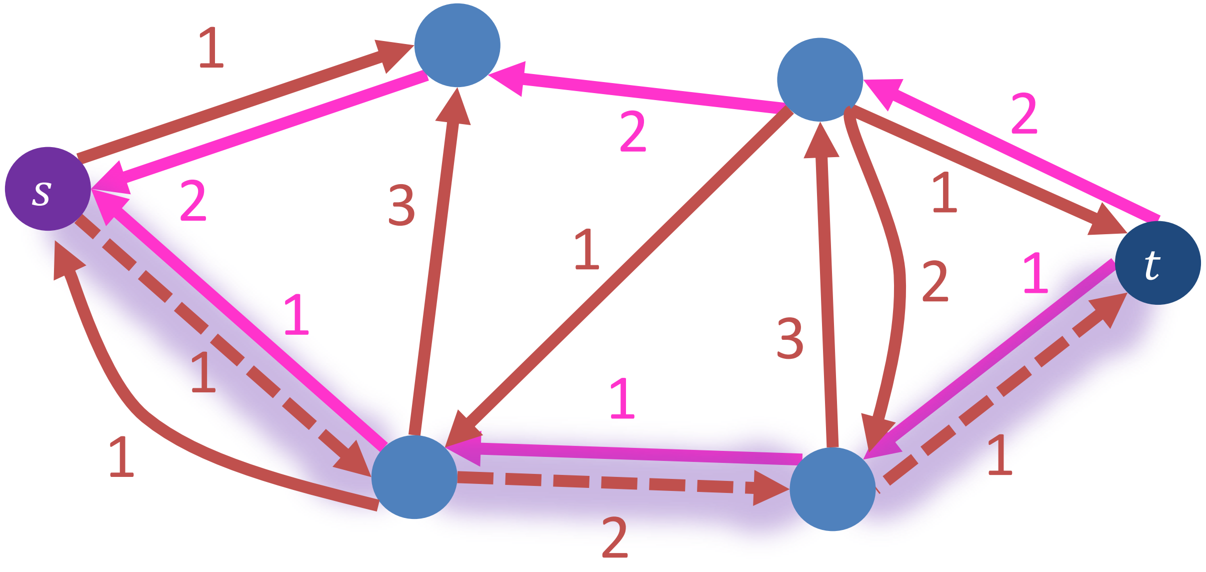

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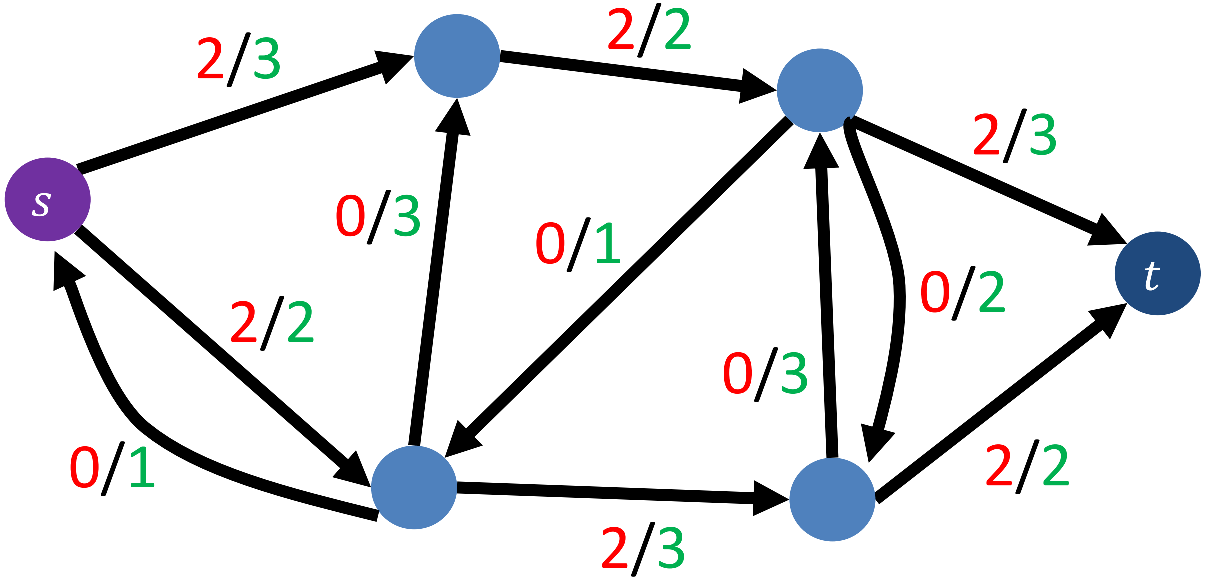
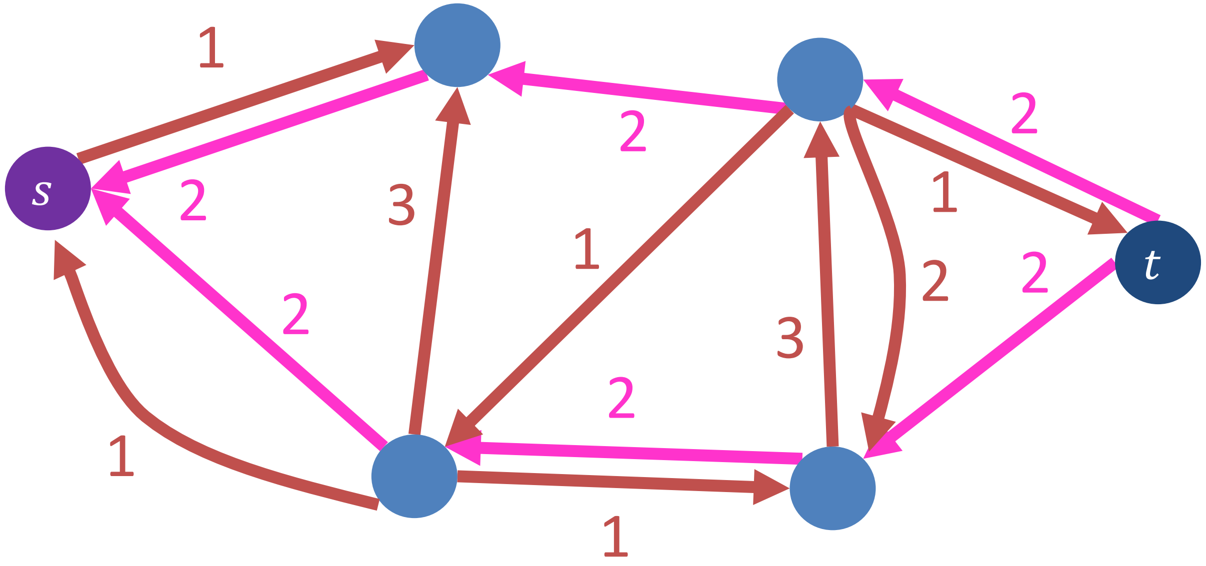
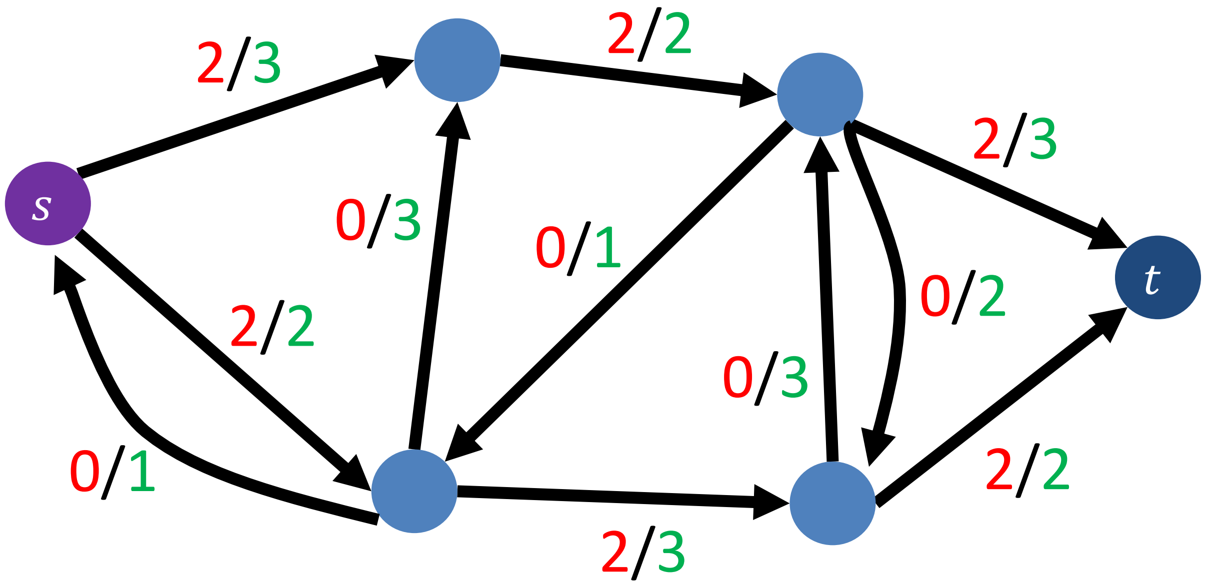


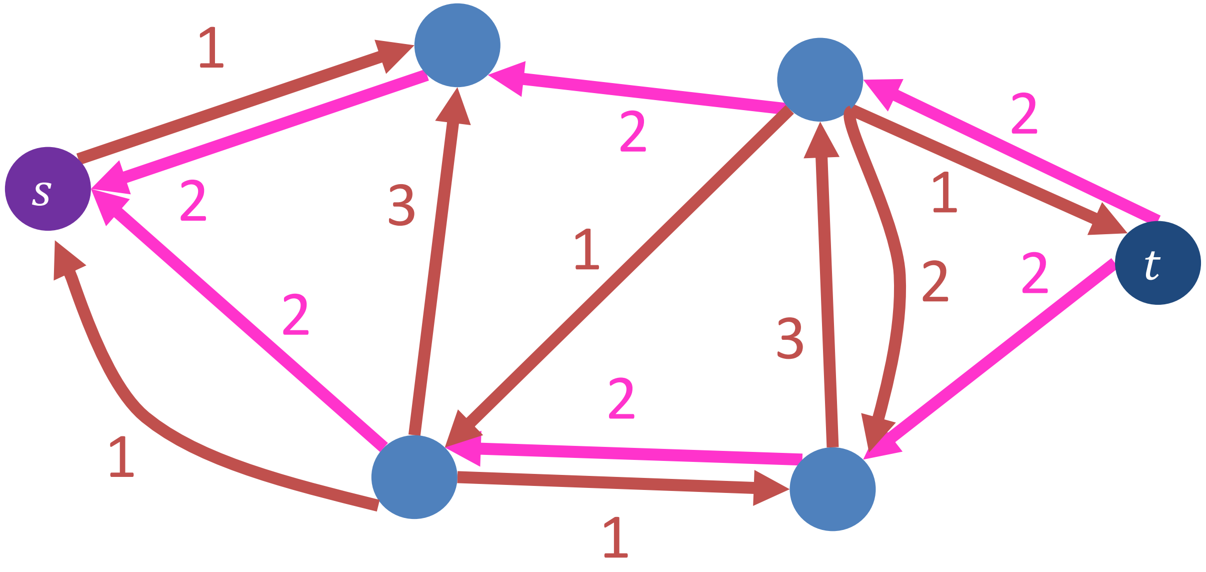


Residual graph G_f

Ford-Fulkerson Example

No more augmenting paths





Residual graph G_f

Maximum flow: 4

Ford-Fulkerson Algorithm - Runtime

Define an **augmenting path** to be a path from $s \to t$ in the residual graph G_f (using edges of non-zero weight)

Overview: Repeatedly add the flow of any augmenting path

Ford-Fulkerson max-flow algorithm:

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Time to find an augmenting path:

Number of iterations of While loop:

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Time to find an augmenting path: BFS: $\Theta(V + E)$

Number of iterations of While loop: $|f|$

$\Theta(E \cdot |f|)$

Why might we loop $|f|$ times?

- Initialize $f(e) = 0$ for all $e \in E$
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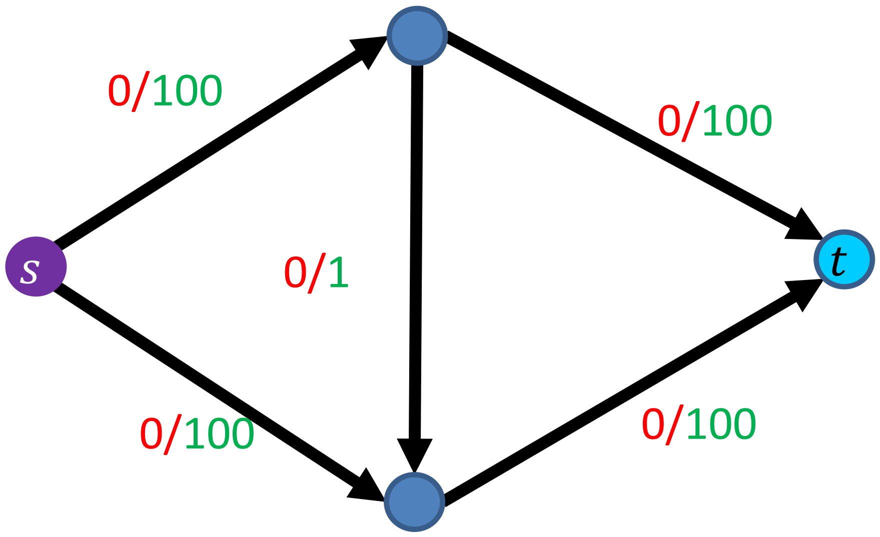

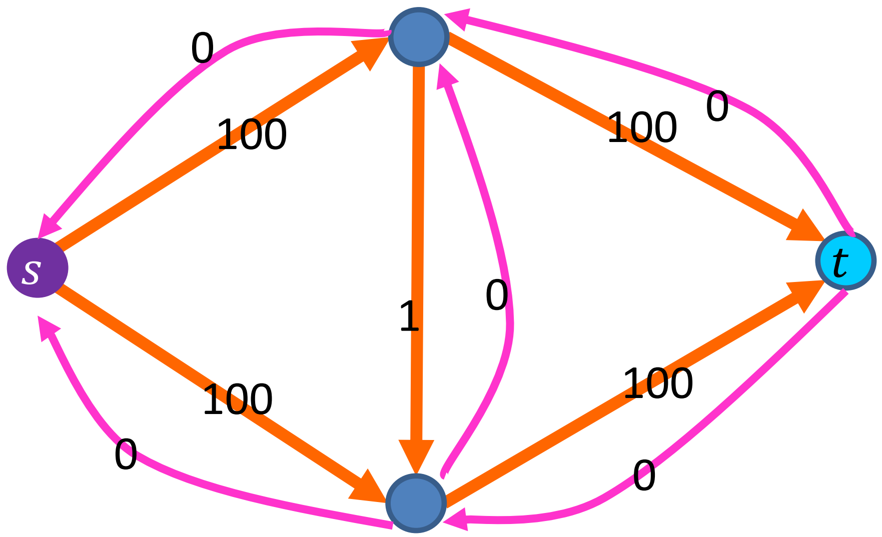

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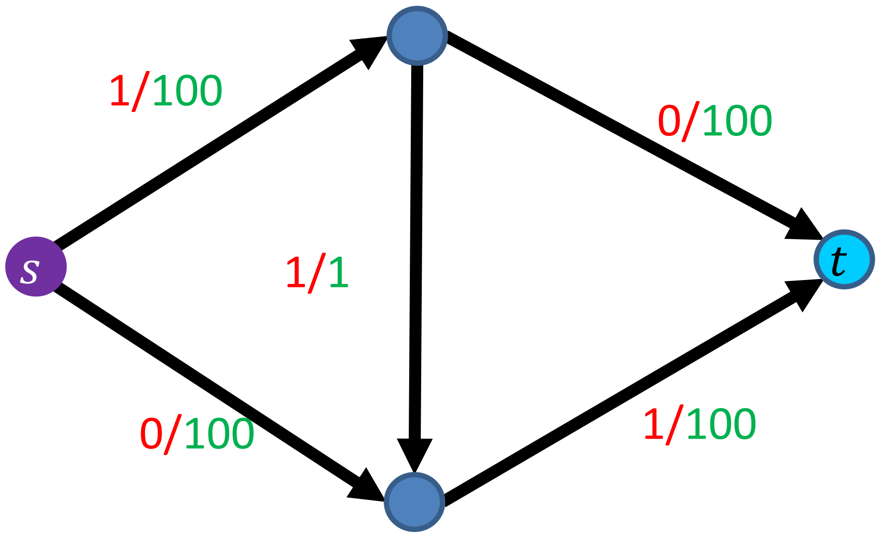

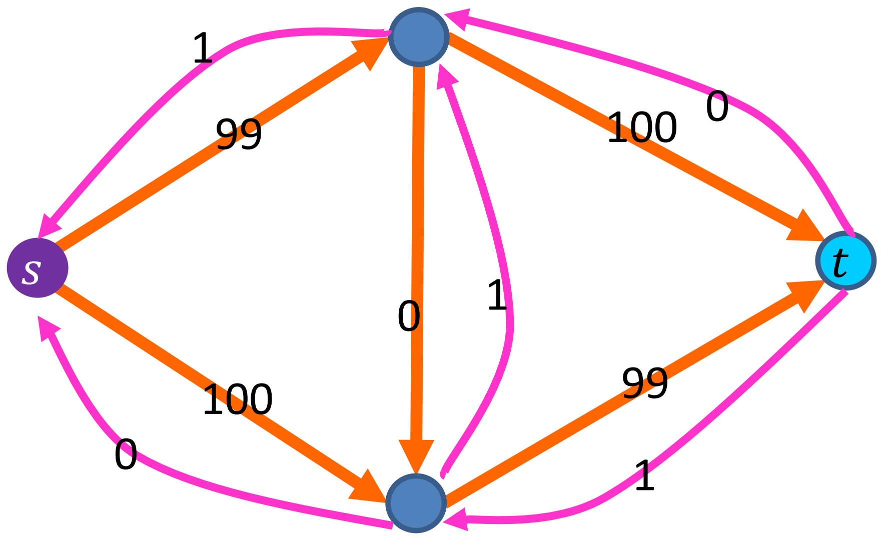

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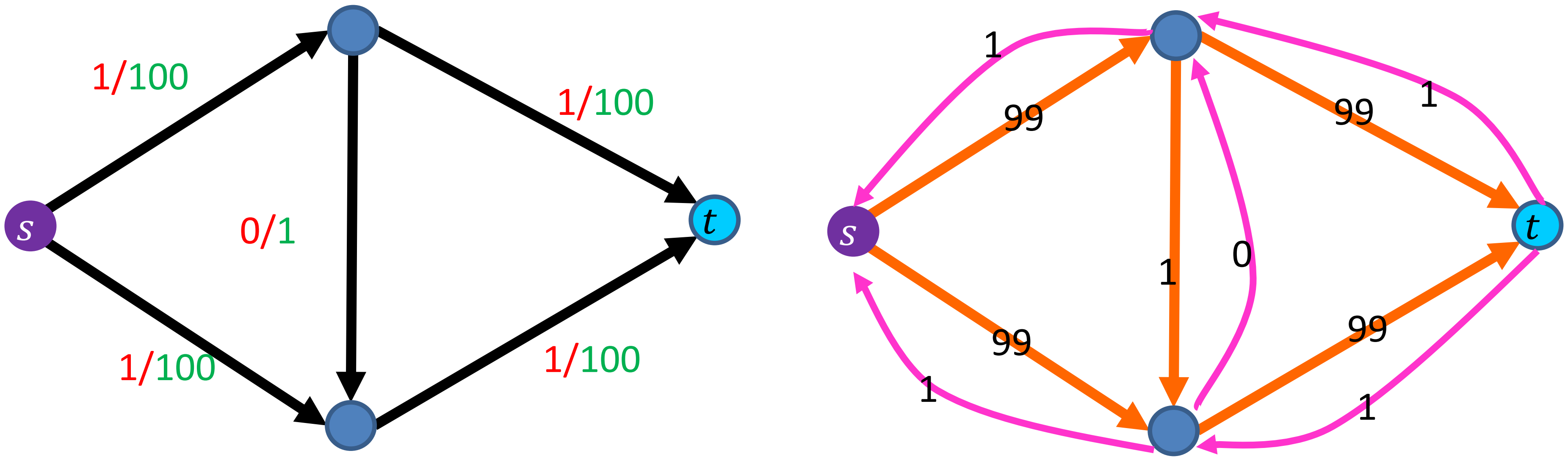

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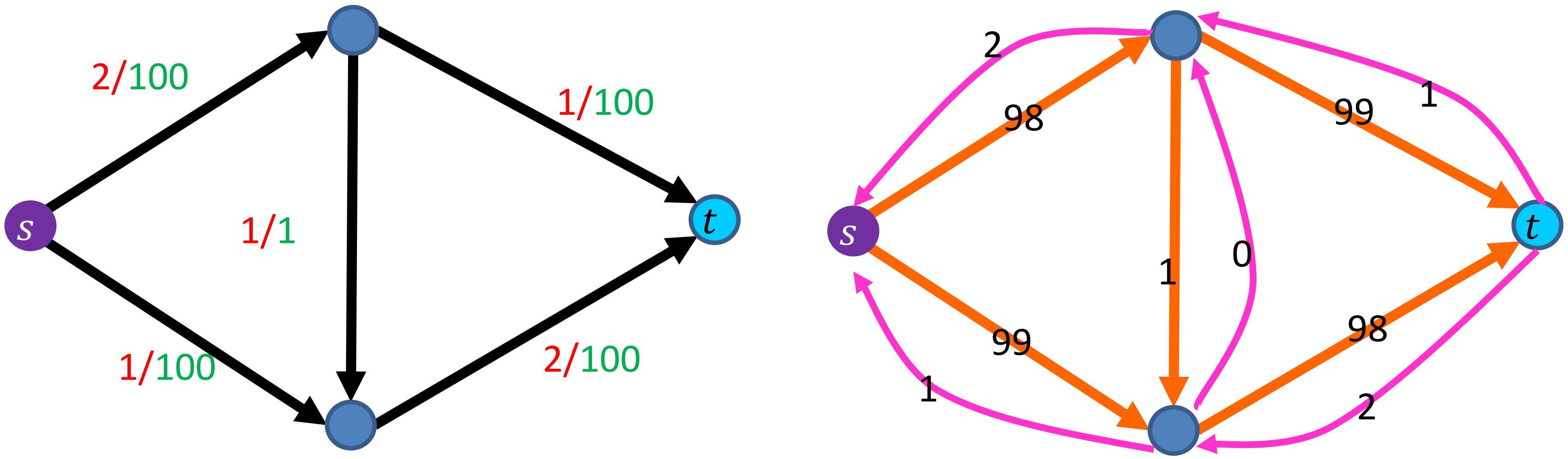
Each time we increase flow by 1

Loop runs 200 times



Can We Avoid this?

- **Edmonds-Karp Algorithm:** choose augmenting path with fewest hops
- **Running time:** $\Theta(\min(|E||f^*|, |V||E|^2)) = O(|V||E|^2)$

Edmonds-Karp max-flow algorithm:

- Initialize $f(e) = 0$ for all $e \in E$
- Construct the residual network G_f
- While there is an augmenting path in G_f, let p be the path with fewest hops:
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Proof: See CLRS (Chapter 26.2)

Can We Avoid this?

- **Edmonds-Karp Algorithm:** choose augmenting path with fewest hops
- **Running time:** $\Theta(\min(|E||f^*|, |V||E|^2))$

How to find this?
Use breadth-first search (BFS)!

Edmonds-Karp = Ford-Fulkerson using BFS to find augmenting path

Edmonds-Karp max-flow algorithm:

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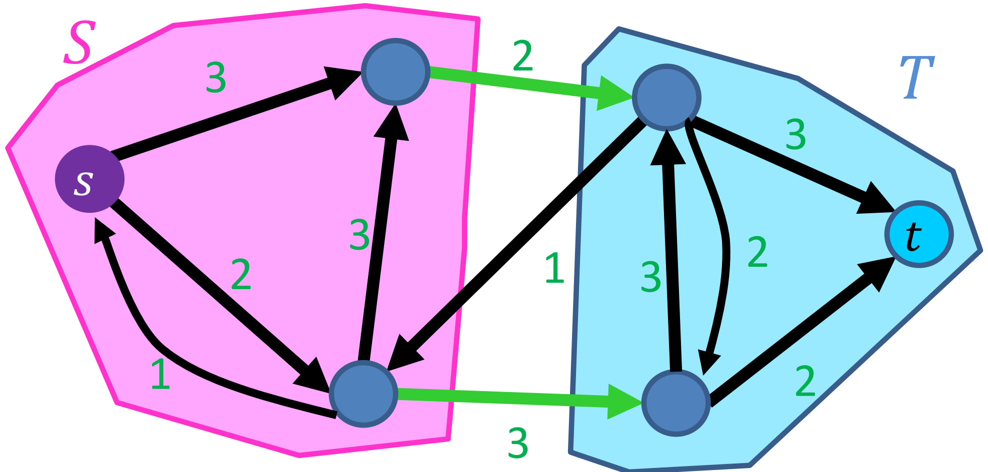
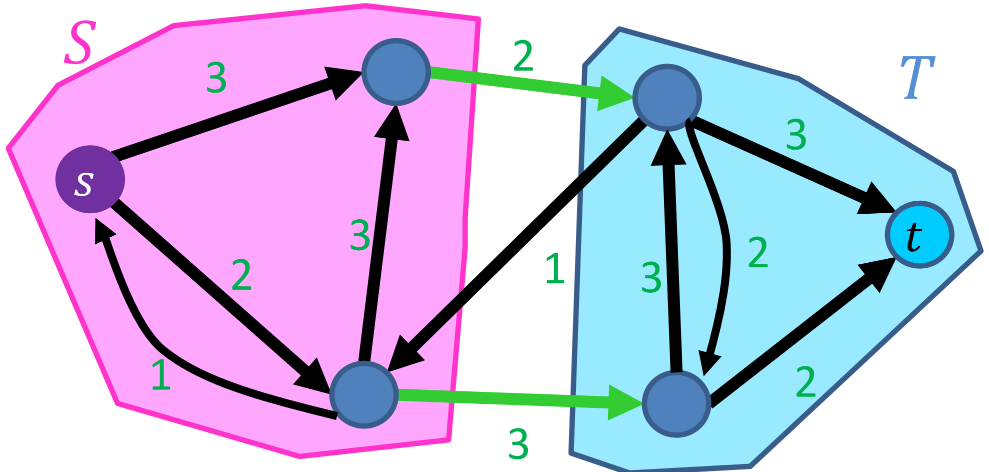
Showing Correctness of Ford-Fulkerson

- Consider cuts which separate s and t
  - Let $s \in S$, $t \in T$, s.t. $V = S \cup T$
- Cost of cut $(S, T) = ||S, T||$
  - Sum **capacities** of edges which go from S to T
  - This example: 5



Maxflow≤MinCut

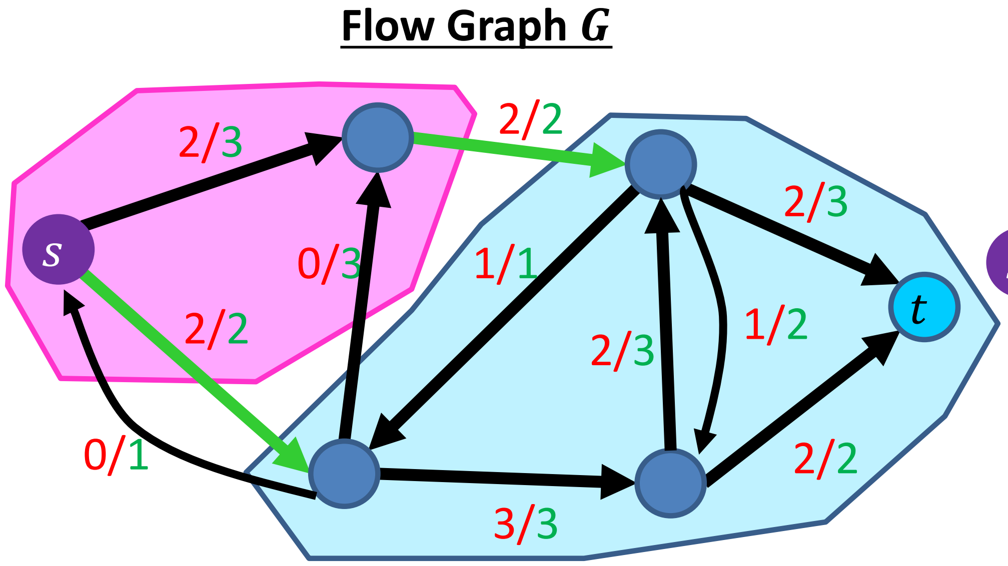
- Max flow upper bounded by any cut separating s and t
- Why? "Conservation of flow"
  - All flow exiting s must eventually get to t
  - To get from s to t, all "tanks" must cross the cut
- Conclusion: If we find the minimum-cost cut, we've found the maximum flow
  - $\max_f |f| \leq \min_{S,T} ||S,T||$

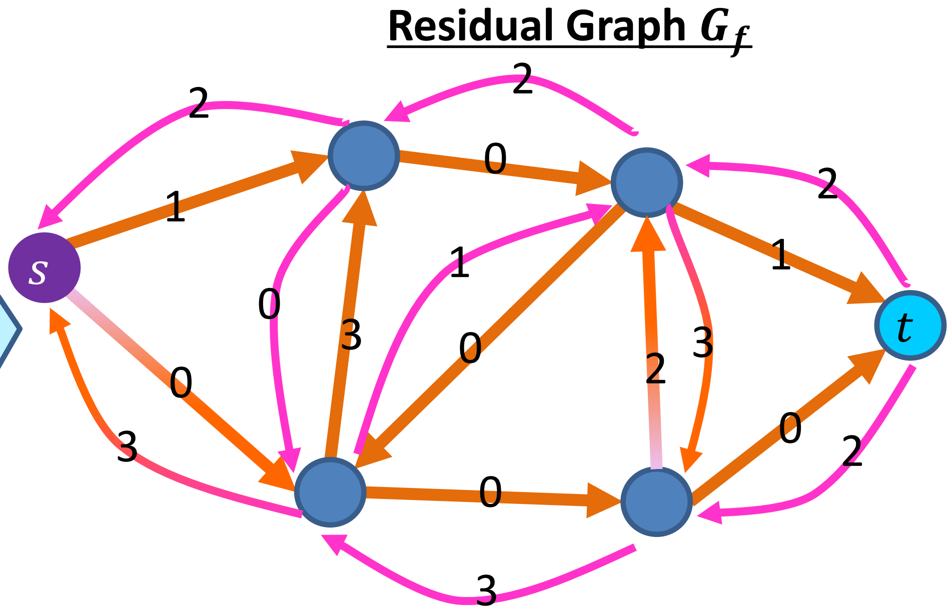


Maxflow/Mincut Theorem

- To show Ford-Fulkerson is correct:
  - Show that when there are no more augmenting paths, there is a cut with cost equal to the flow
- Conclusion: the maximum flow through a network matches the minimum-cost cut
  - $\max_{f}|f| = \min_{S,T} ||S,T||$
- Duality
  - When we've maximized max flow, we've minimized min cut (and vice-versa), so we can check when we've found one by finding the other

Example: Maxflow/Mincut





$|f| = 4$

$||S, T|| = 4$

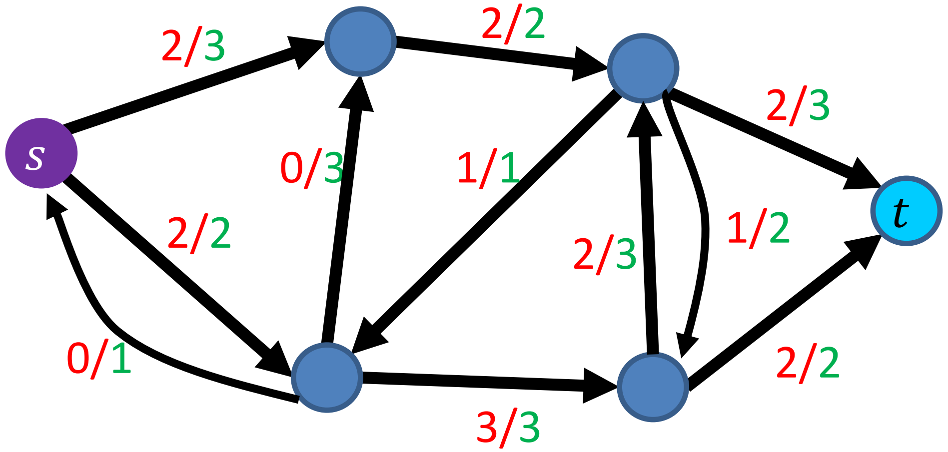
No Augmenting Paths

Idea: When there are no more augmenting paths, there exists a cut in the graph with cost matching the flow

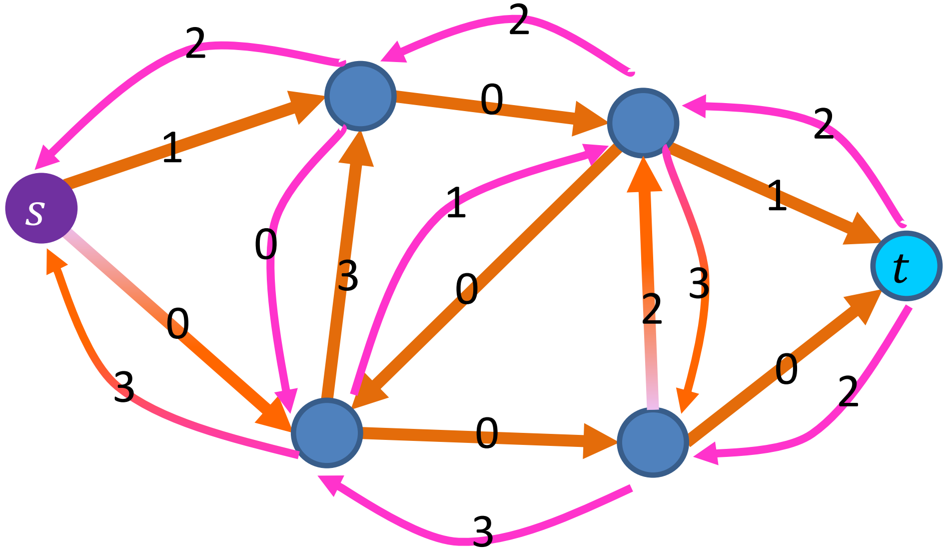
Proof: Maxflow/Mincut Theorem

- If $|f|$ is a max flow, then G_f has no augmenting path
  - Otherwise, use that augmenting path to "push" more flow

Flow Graph G



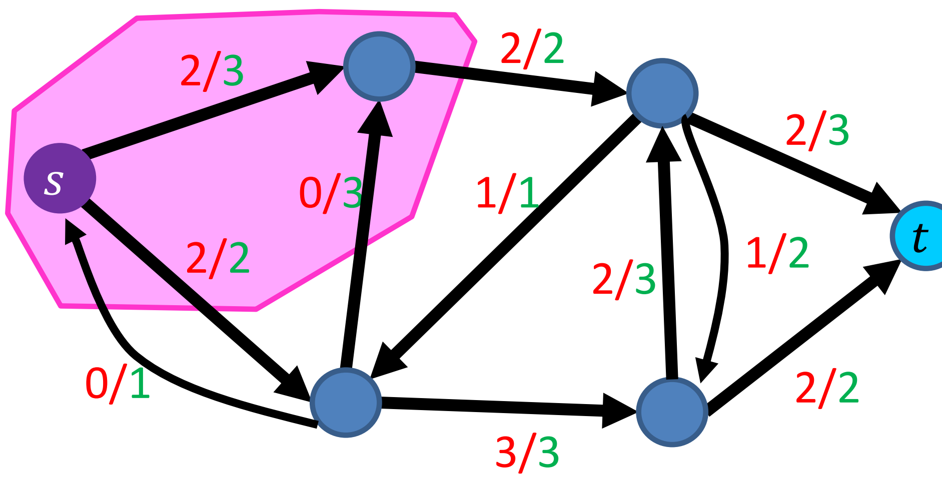
Residual Graph G_f



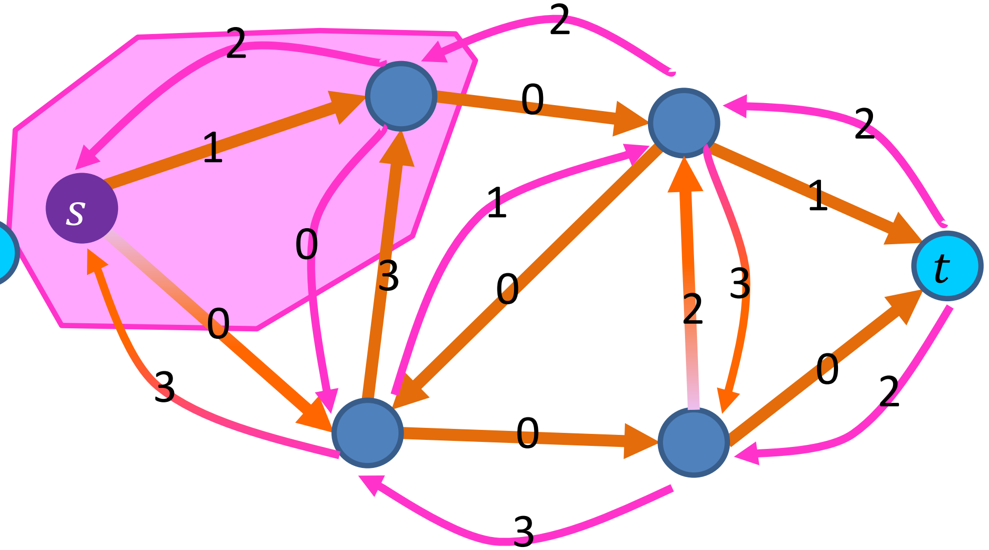
Proof: Maxflow/Mincut Theorem

- If $|f|$ is a max flow, then G_f has no augmenting path
  - Otherwise, use that augmenting path to “push” more flow
- Define $S =$ nodes reachable from source node s by positive-weight edges in the residual graph
  - $T = V - S$
  - S separates s, t (otherwise there’s an augmenting path)

Flow Graph G



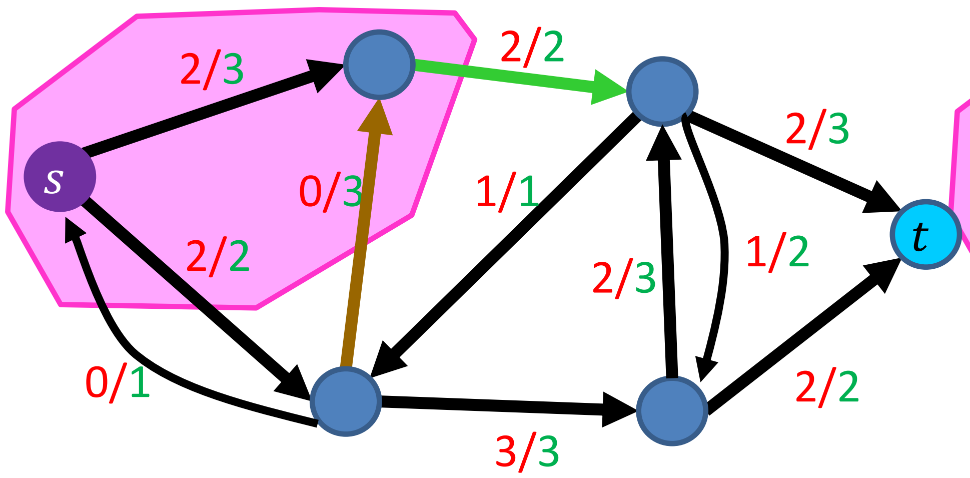
Residual Graph G_f



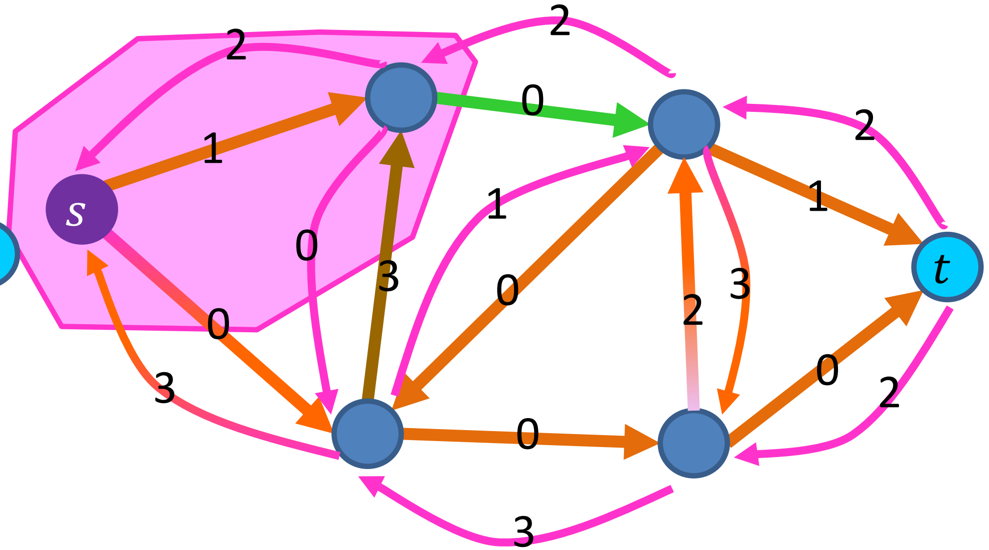
Proof: Maxflow/Mincut Theorem

- To show: $||S, T|| = |f|$
  - Weight of the cut matches the flow across the cut
- Consider edge (u, v) with $u \in S$, $v \in T$
  - $f(u, v) = c(u, v)$, because otherwise $w(u, v) > 0$ in G_f, which would mean $v \in S$
- Consider edge (y, x) with $y \in T$, $x \in S$
  - $f(y, x) = 0$, because otherwise the back edge $w(y, x) > 0$ in G_f, which would mean $x \in S$

Flow Graph G



Residual Graph G_f



Proof Summary

1. The flow $|f|$ of G is upper-bounded by the sum of capacities of edges crossing any cut separating source s and sink t

2. When Ford-Fulkerson terminates, there are no more augmenting paths in G_f

3. When there are no more augmenting paths in G_f then we can define a cut $S =$ nodes reachable from source node s by positive-weight edges in the residual graph

4. The sum of edge capacities crossing this cut must match the flow of the graph

5. Therefore this flow is maximal

Other Maxflow algorithms

- **Ford-Fulkerson**
  - $\boldsymbol{\Theta}(\boldsymbol{E}|\boldsymbol{f}|)$
- **Edmonds-Karp**
  - $\boldsymbol{\Theta}(\boldsymbol{E^2V})$
- Push-Relabel (Tarjan)
  - $\Theta(EV^2)$
- Faster Push-Relabel (also Tarjan)
  - $\Theta(V^3)$