CS4102 Algorithms

Spring 2020

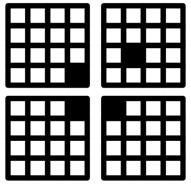
Today's Keywords

- Reductions
- Bipartite Matching

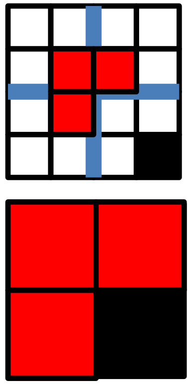
CLRS Readings

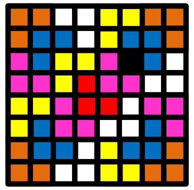
- Chapter 34

Divide and Conquer*



- **Divide**:
  - Break the problem into multiple subproblems, each smaller instances of the original
- **Conquer**:
  - If the suproblems are "large":
    - Solve each subproblem recursively
  - If the subproblems are "small":
    - Solve them directly (base case)
- **Combine**:
  - Merge together solutions to subproblems





*CLRS Chapter 4

Dynamic Programming

- Requires Optimal Substructure
  - – Solution to larger problem contains the solutions to smaller ones
- Idea:
  1. Identify recursive structure of the problem
  2. Select a good order for solving subproblems
     - Usually smallest problem first

Greedy Algorithms

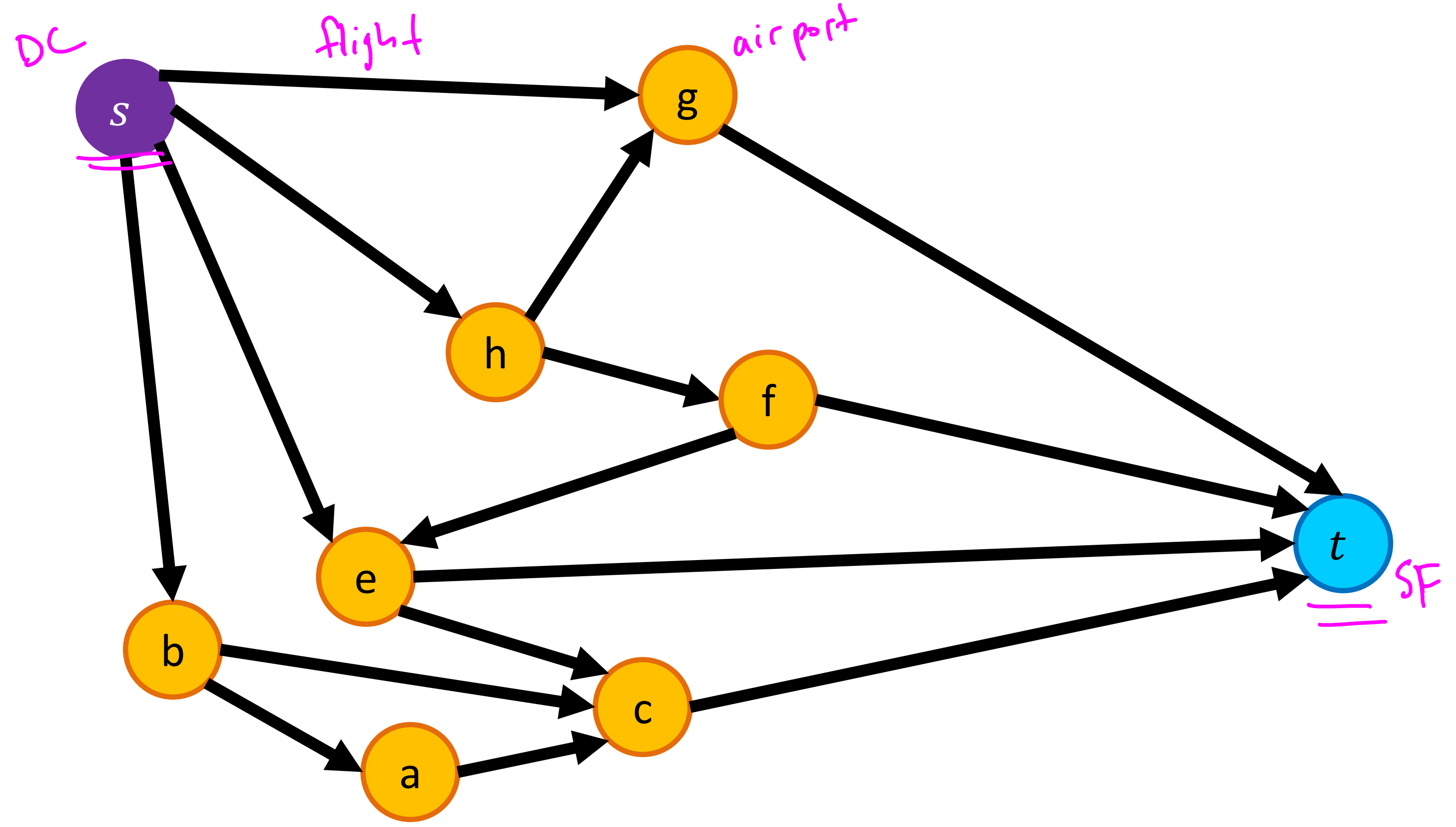
- Require Optimal Substructure
  - Solution to larger problem contains the solution to a smaller one
  - Only one subproblem to consider!
- Idea:
  1. Identify a greedy choice property
     - How to make a choice guaranteed to be included in some optimal solution
  2. Repeatedly apply the choice property until no subproblems remain

So far

- Divide and Conquer, Dynamic Programming, Greedy
  - Take an instance of *Problem A*, relate it to smaller instances of *Problem A*
- Next: Reductions
  - Take an instance of *Problem A*, relate it to an instance of ***Problem B***

Edge-Disjoint Paths

Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no edges



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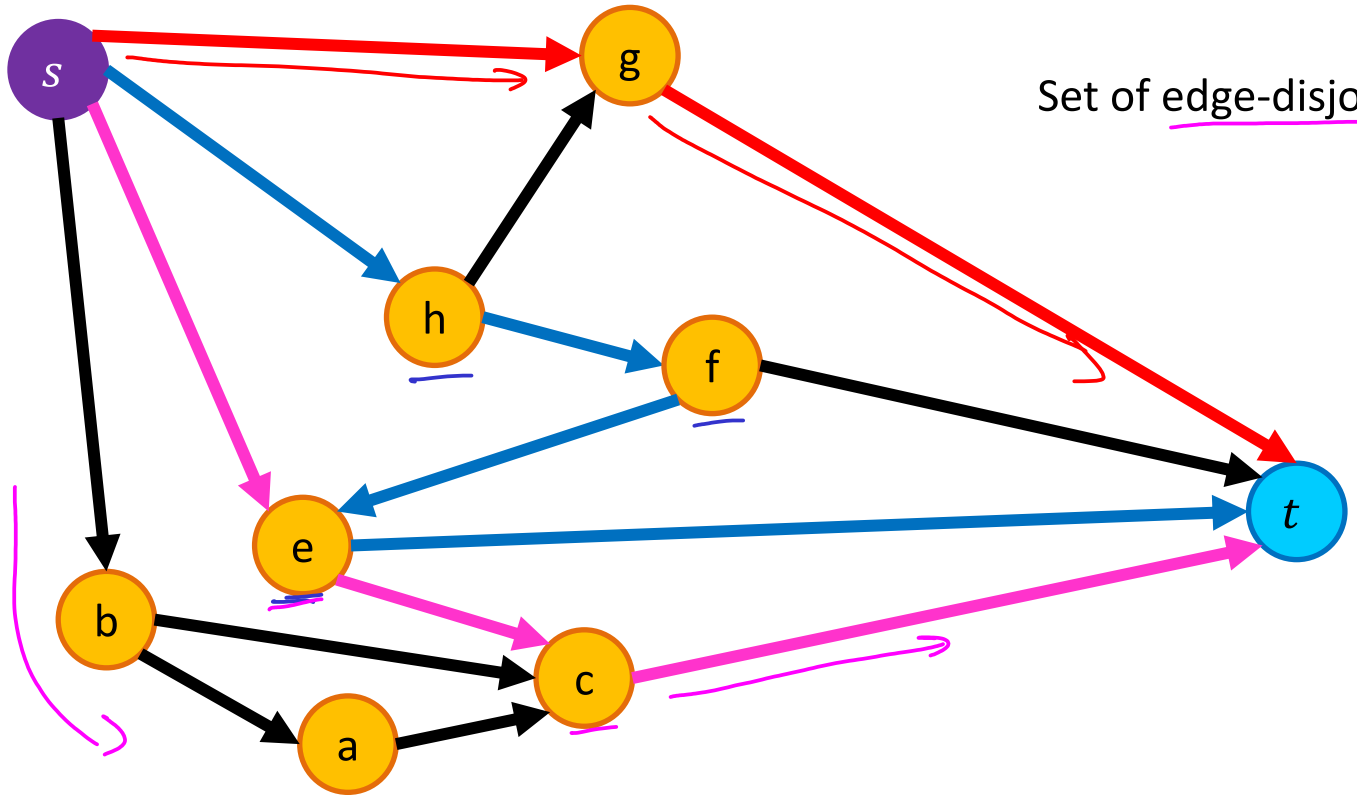

Set of edge-disjoint paths of size 3

Edge-Disjoint Paths

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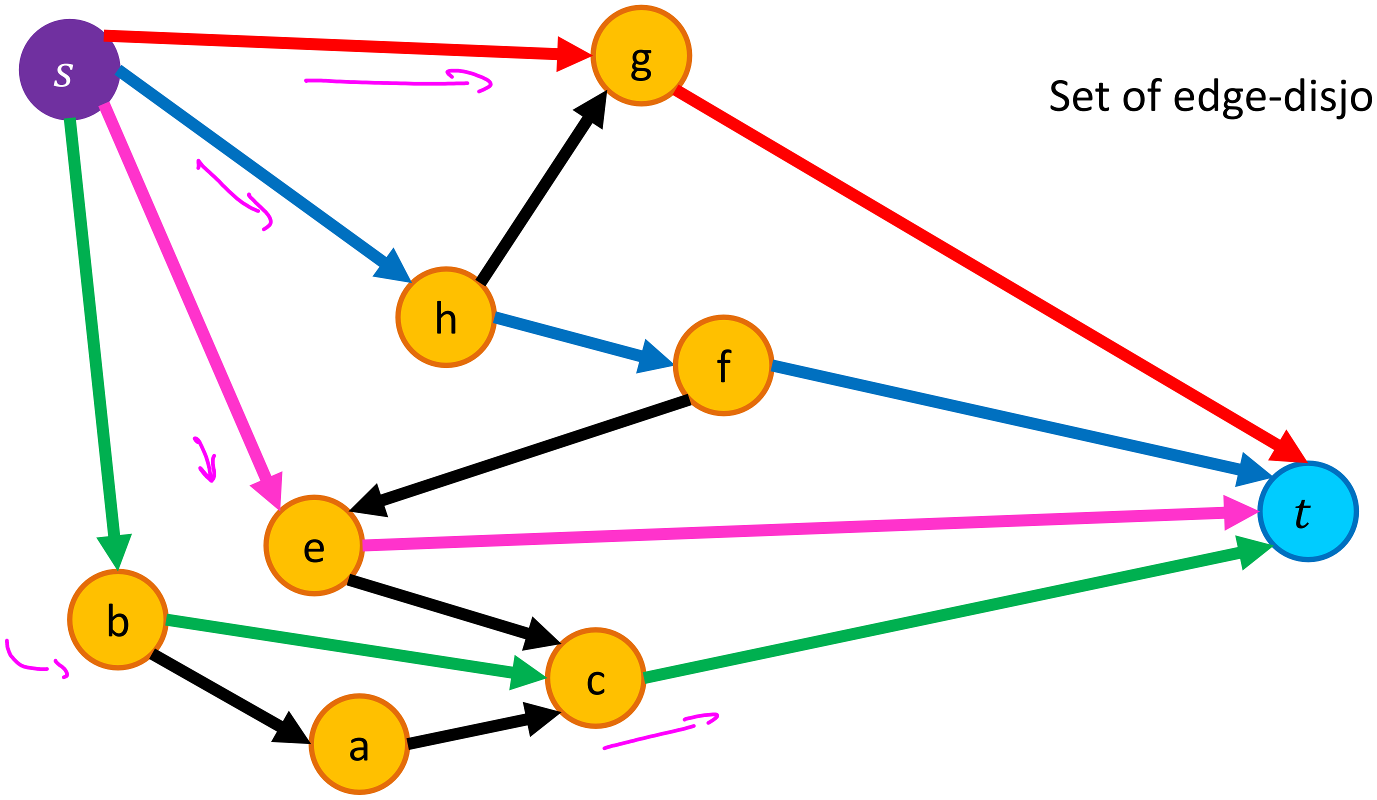
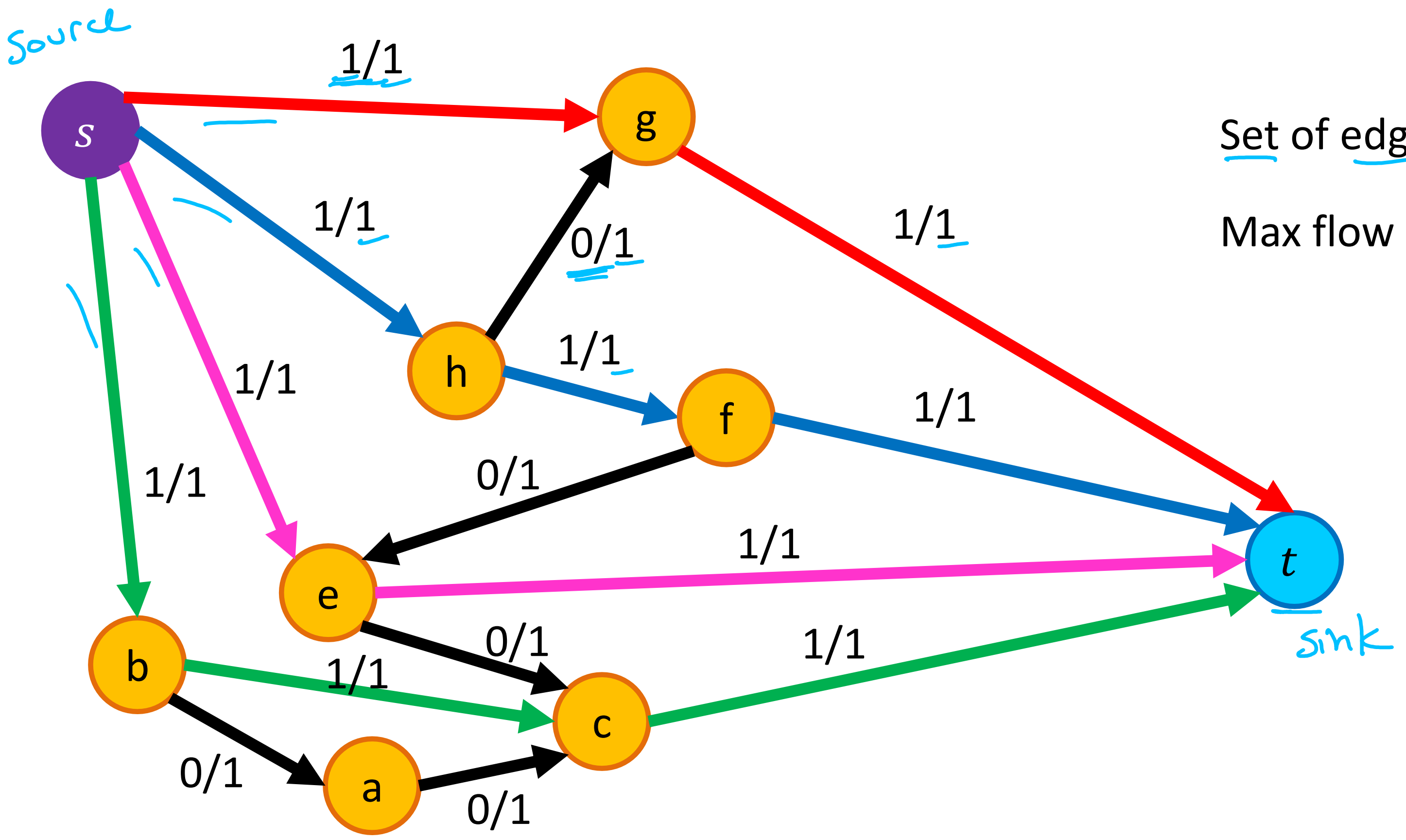


Set of edge-disjoint paths of size 4

Edge-Disjoint Paths Algorithm

Make s and t the source and sink, give each edge capacity 1, find the max flow.

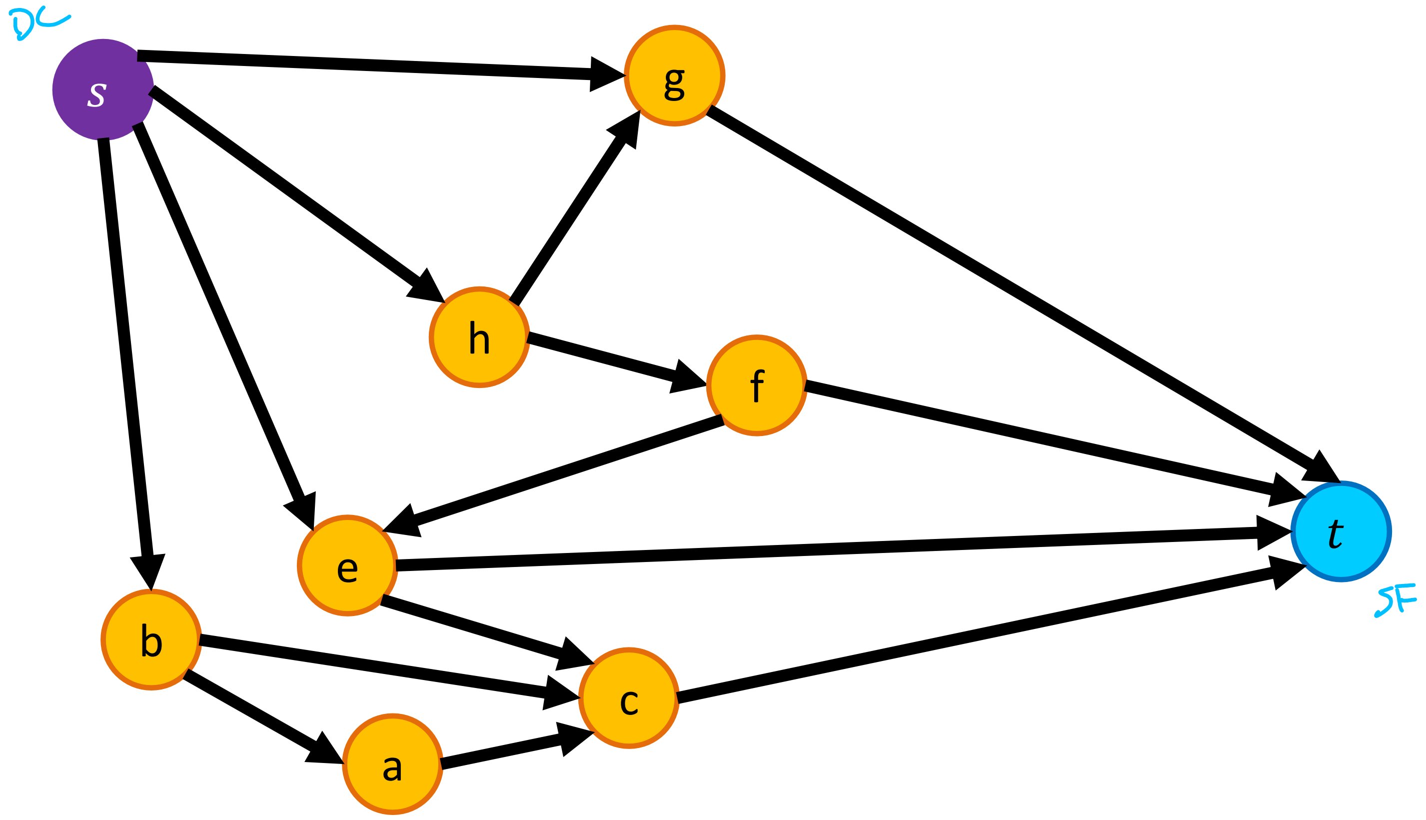


Set of edge-disjoint paths of size 4

Max flow = 4

Vertex-Disjoint Paths

Given a graph $G = (V, E)$, a start node s and a destination node t, give the maximum number of paths from s to t which share no vertices



Vertex-Disjoint Paths

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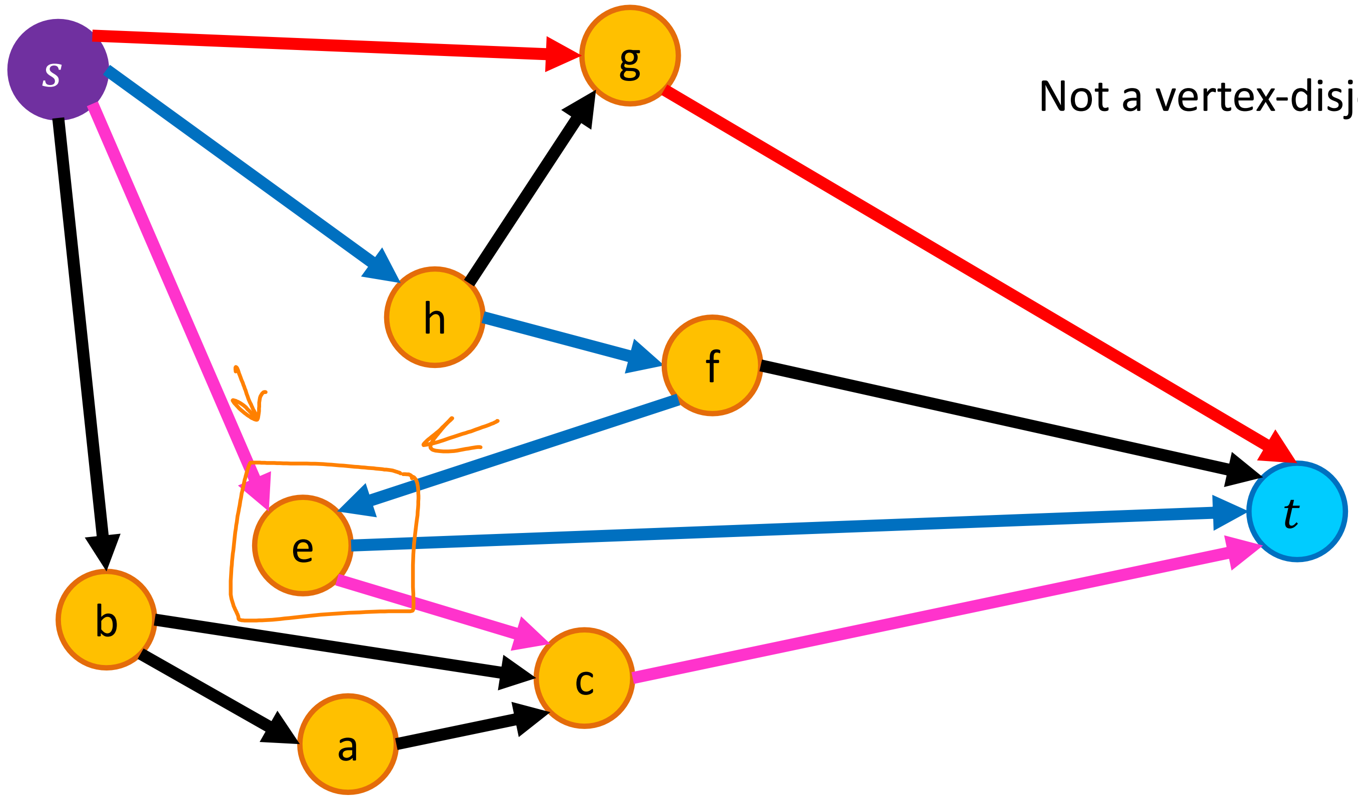
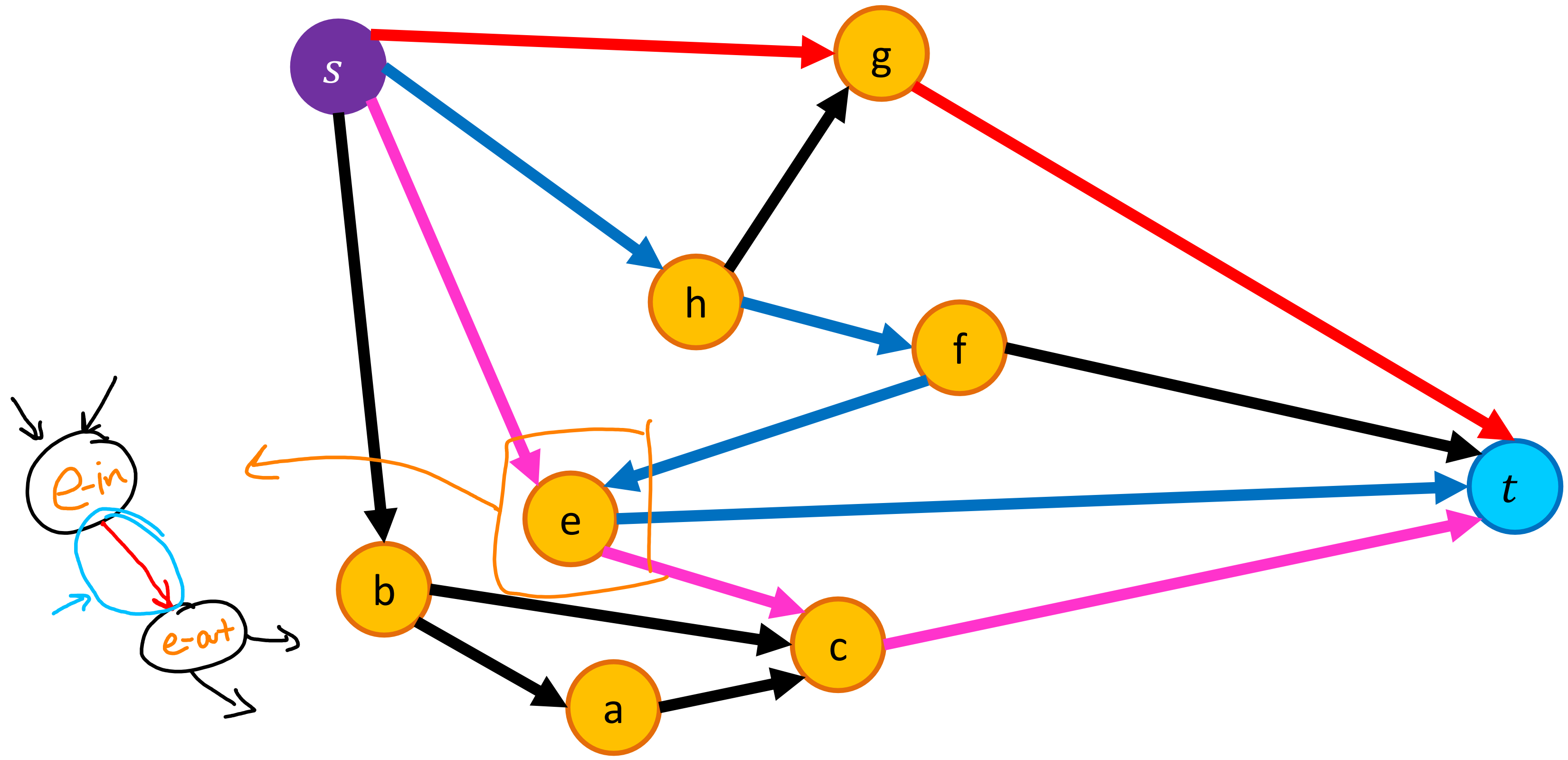


Not a vertex-disjoint path!

Vertex-Disjoint Paths Algorithm

Idea: Convert an instance of the vertex-disjoint paths problem into an instance of edge-disjoint paths

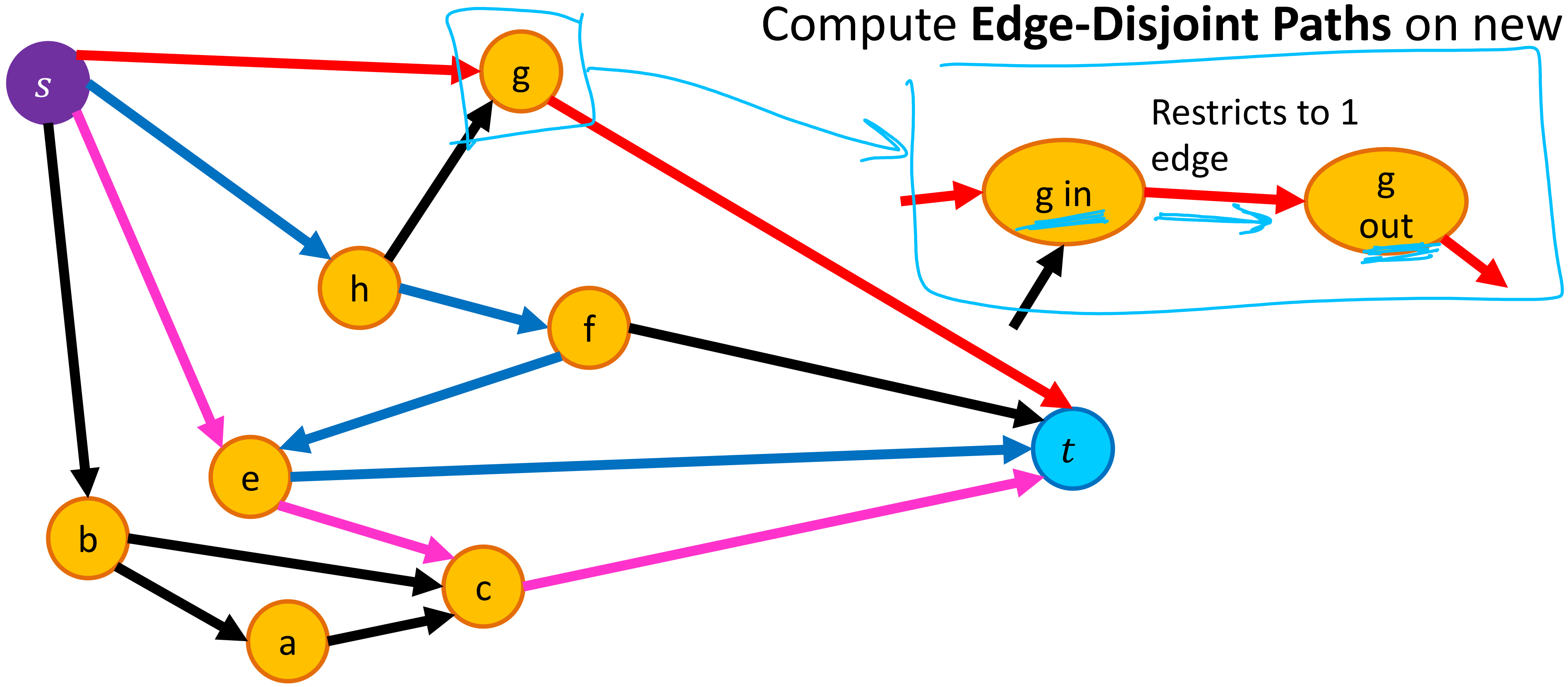


Vertex-Disjoint Paths Algorithm

Idea: Convert an instance of the vertex-disjoint paths problem into an instance of edge-disjoint paths

Make two copies of each node, one connected to incoming edges, the other to outgoing edges

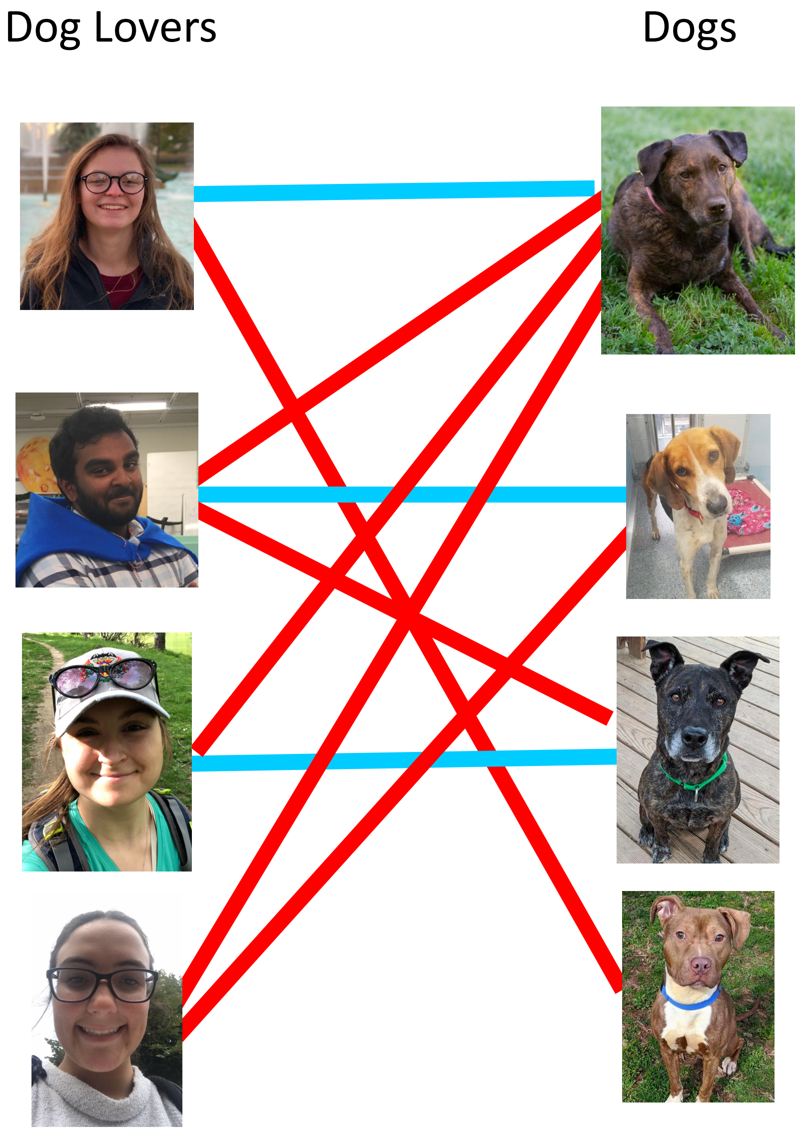
Compute **Edge-Disjoint Paths** on new graph



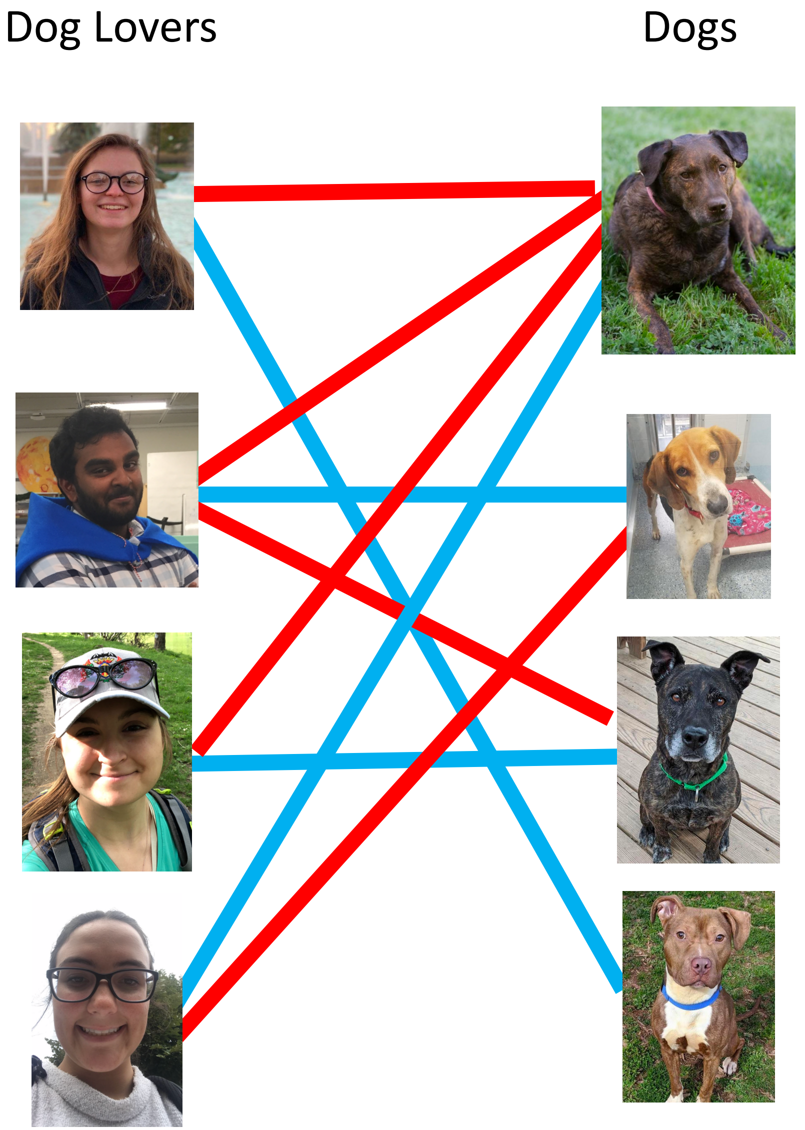
Maximum Bipartite Matching

Dog Lovers　　　　Dogs

Maximum Bipartite Matching



Maximum Bipartite Matching



Maximum Bipartite Matching

Given a graph $G = (L, R, E)$

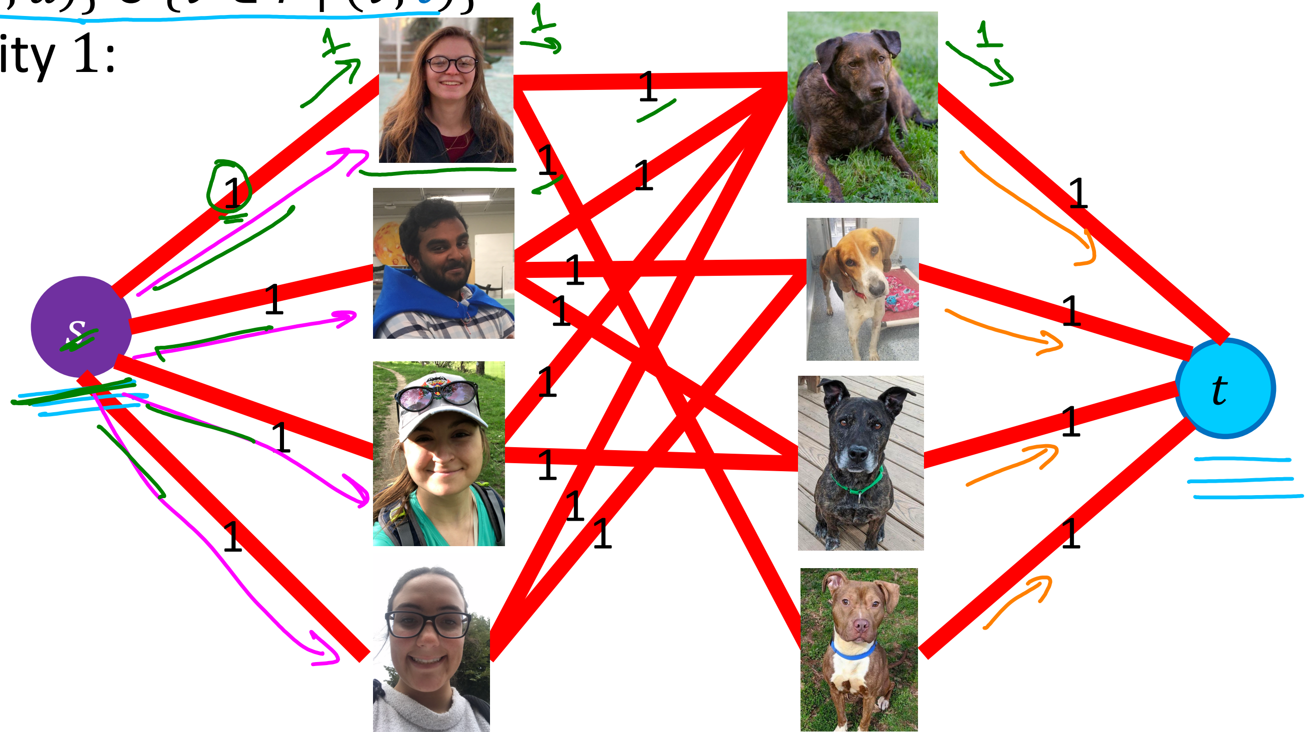
a set of left nodes, right nodes, and edges between left and right

Find the largest set of edges $M \subseteq E$ such that each node $u \in L$ or $v \in R$ is incident to at most one edge.

Maximum Bipartite Matching Using Max Flow

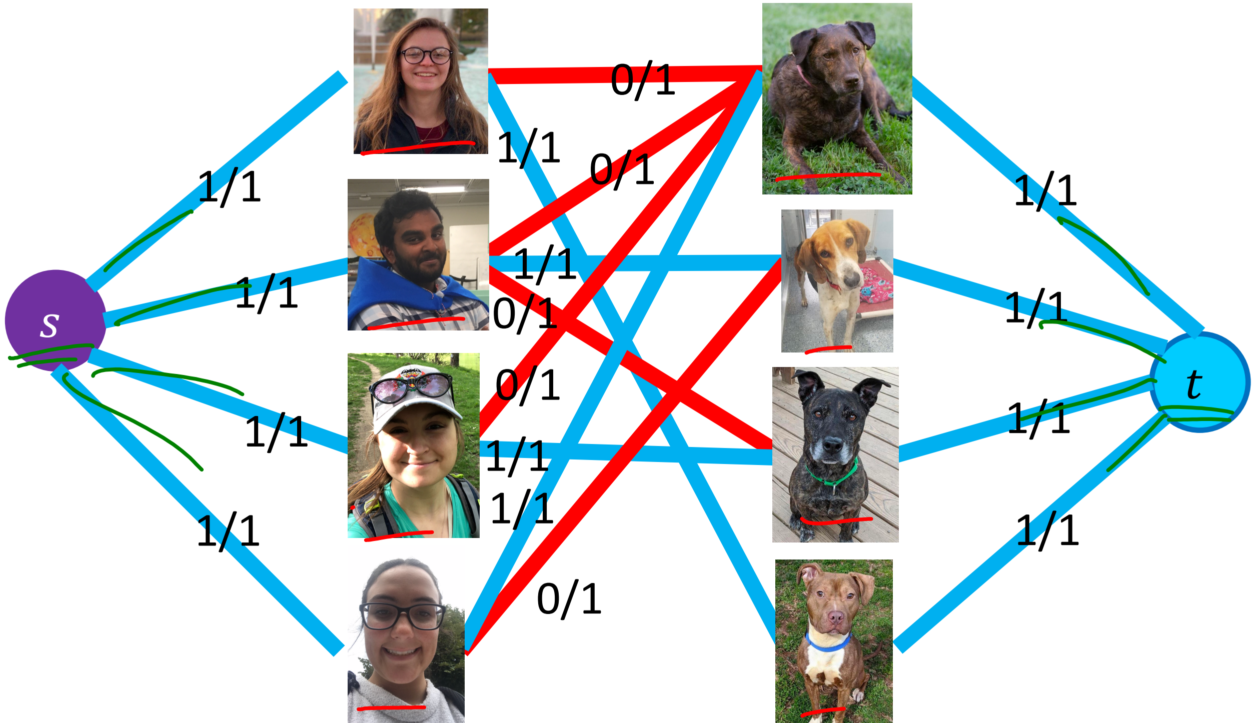
Make $G = (L, R, E)$ a flow network $G' = (V', E')$ by:

- Adding in a source and sink to the set of nodes:
  - $V' = L \cup R \cup \{s, t\}$
- Adding an edge from source to L and from R to sink:
  - $E' = E \cup \{u \in L \mid (s, u)\} \cup \{v \in r \mid (v, t)\}$
- Make each edge capacity 1:
  - $\forall e \in E', c(e) = 1$



Maximum Bipartite Matching Using Max Flow

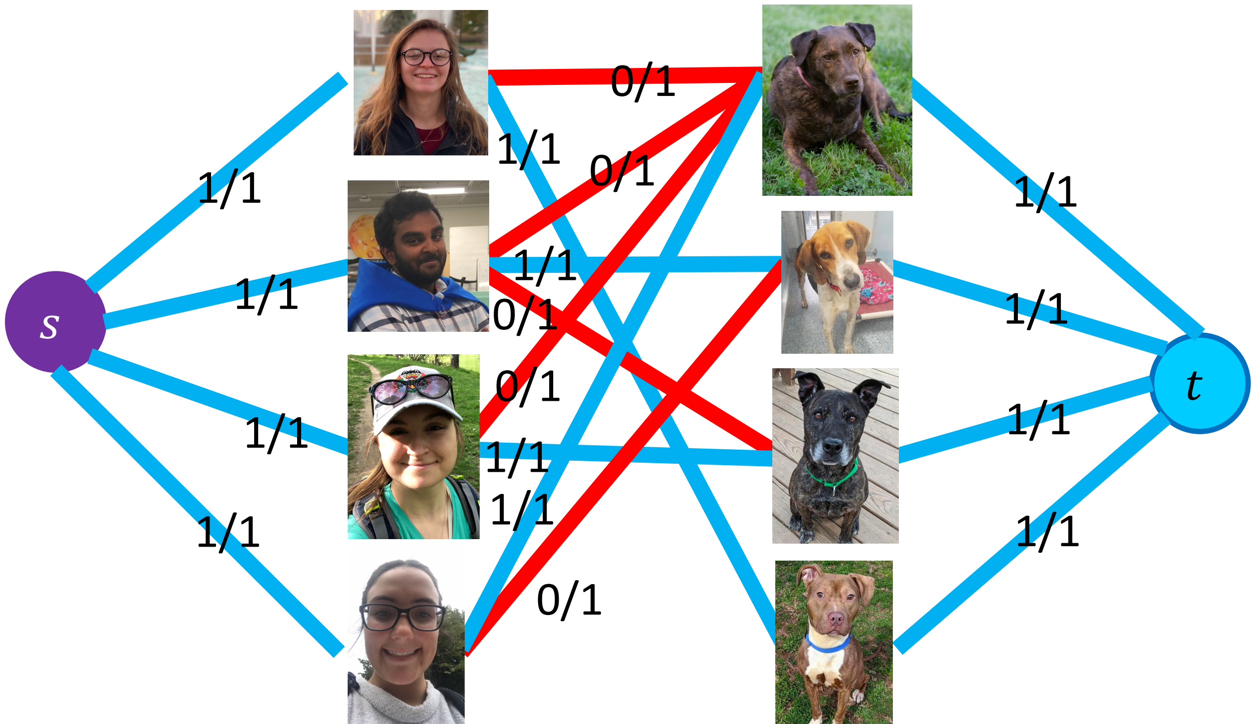
1. Make G into G' — $\Theta(L+R)$
2. Compute Max Flow on G' — $FF = O(E \cdot |f|)$ — $\Theta(E \cdot V)$ — $|f| \le \min(L,R)$
3. Return M as all "middle" edges with flow 1 — $\Theta(L+R)$



Maximum Bipartite Matching Using Max Flow

$\Theta(E \cdot V)$

1. Make G into G' $\Theta(L + R)$
2. Compute Max Flow on G' $\Theta(E \cdot V)$ $|f| \leq L$
3. Return M as all "middle" edges with flow 1 $\Theta(L + R)$



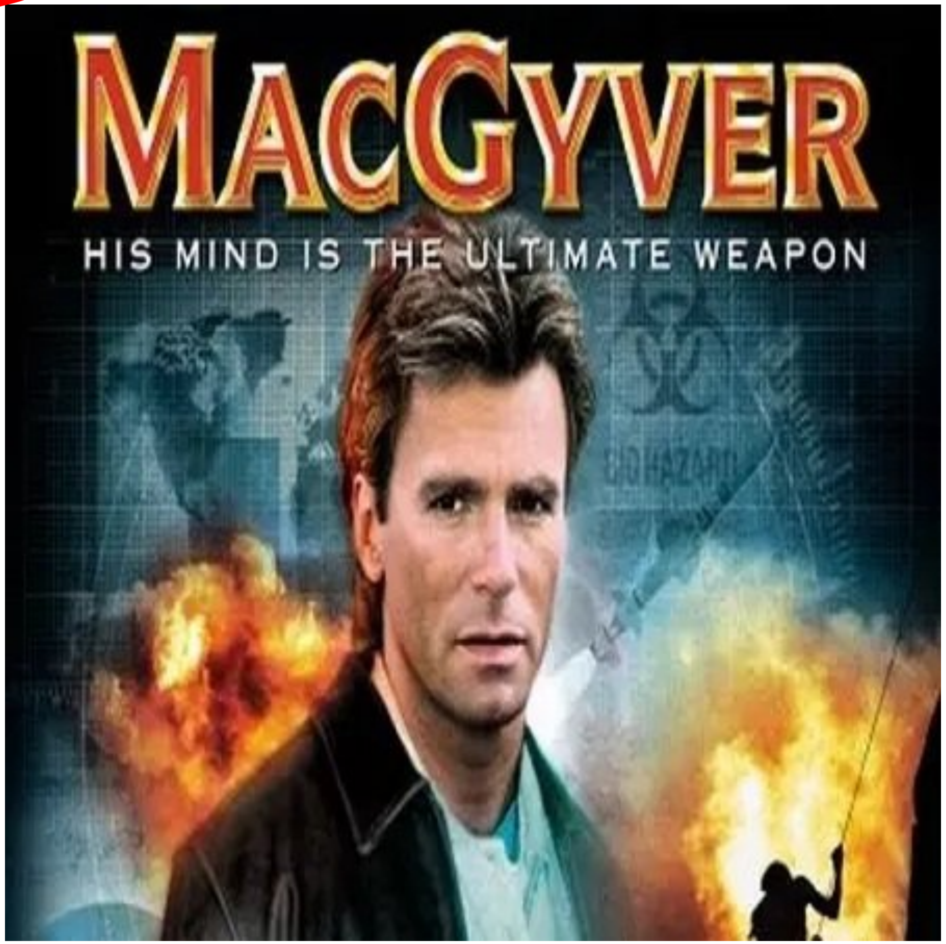
Reductions

- Algorithm technique of supreme ultimate power
- Convert instance of problem A to an instance of Problem B
- Convert solution of problem B back to a solution of problem A

Reductions

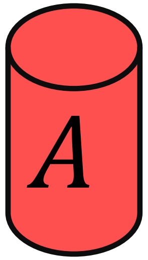
Shows how two different problems relate to each other

MOVIE TIME!



MacGyver's Reduction

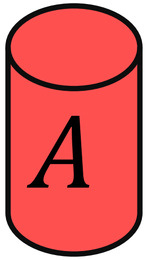
Problem we don't know how to solve



Opening a door

MacGyver's Reduction

Problem we don't know how to solve



Opening a door

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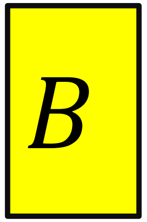
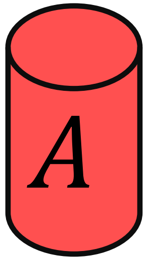


Lighting a fire

MacGyver's Reduction

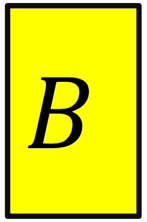
Problem we don't know how to solve



Opening a door

Aim duct at door, insert keg

Problem we do know how to solve



Lighting a fire

how?

MacGyver's Reduction

Problem we don't know how to solve

A

Opening a door

Aim duct at door, insert keg

Problem we do know how to solve

B

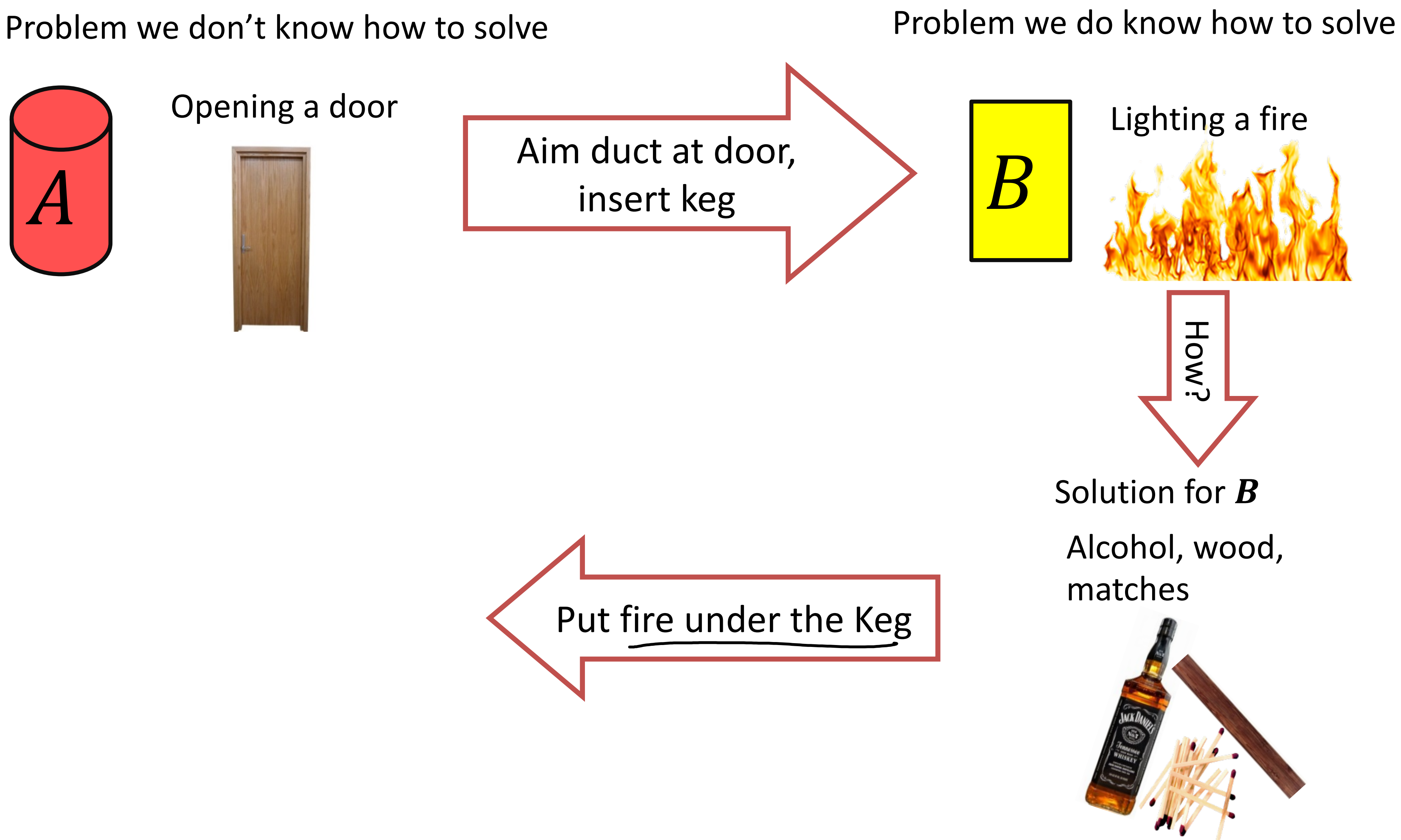
Lighting a fire

How?

Solution for $\boldsymbol{B}$

Alcohol, wood, matches

MacGyver's Reduction



MacGyver's Reduction

Problem we don't know how to solve

A

Opening a door

Aim duct at door, insert keg

Problem we do know how to solve

B

Lighting a fire

How?

Solution for $\boldsymbol{B}$

Alcohol, wood, matches

Put fire under the Keg

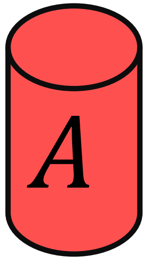
Solution for $\boldsymbol{A}$

Keg cannon

battering ram

MacGyver's Reduction

Problem we don't know how to solve



Opening a door

Aim duct at door, insert keg

Put fire under the Keg

Reduction

Problem we do know how to solve

B

Lighting a fire

How?

Solution for $\boldsymbol{B}$

Alcohol, wood, matches

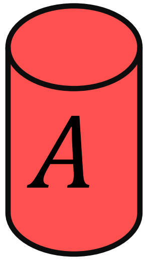
Solution for $\boldsymbol{A}$

Keg cannon battering ram

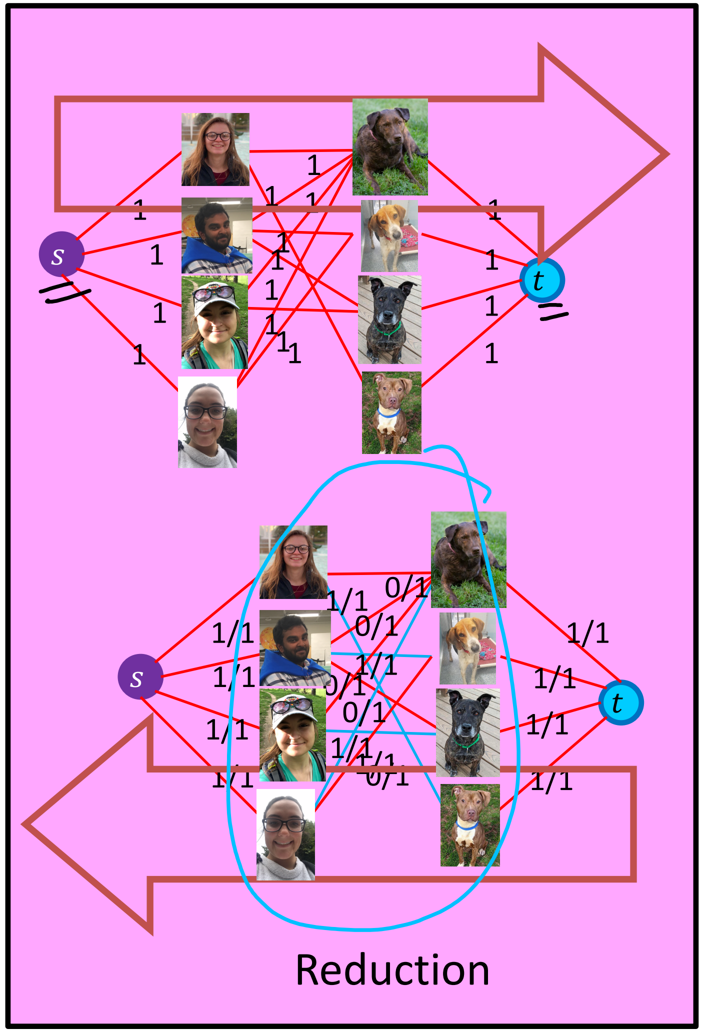
Bipartite Matching Reduction

Problem we don't know how to solve

Bipartite Matching

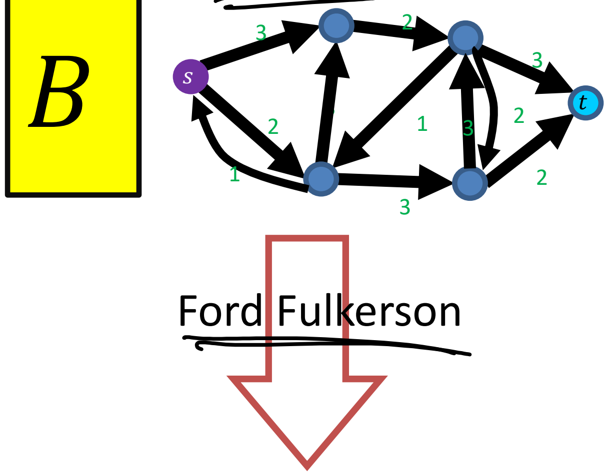


Solution for A

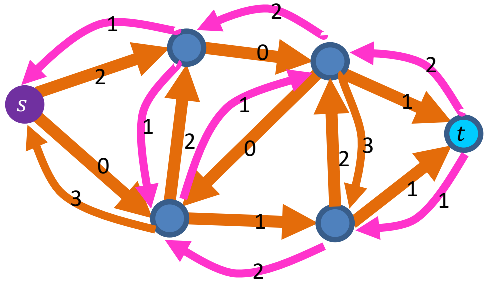


Problem we do know how to solve

Max Flow



Solution for B

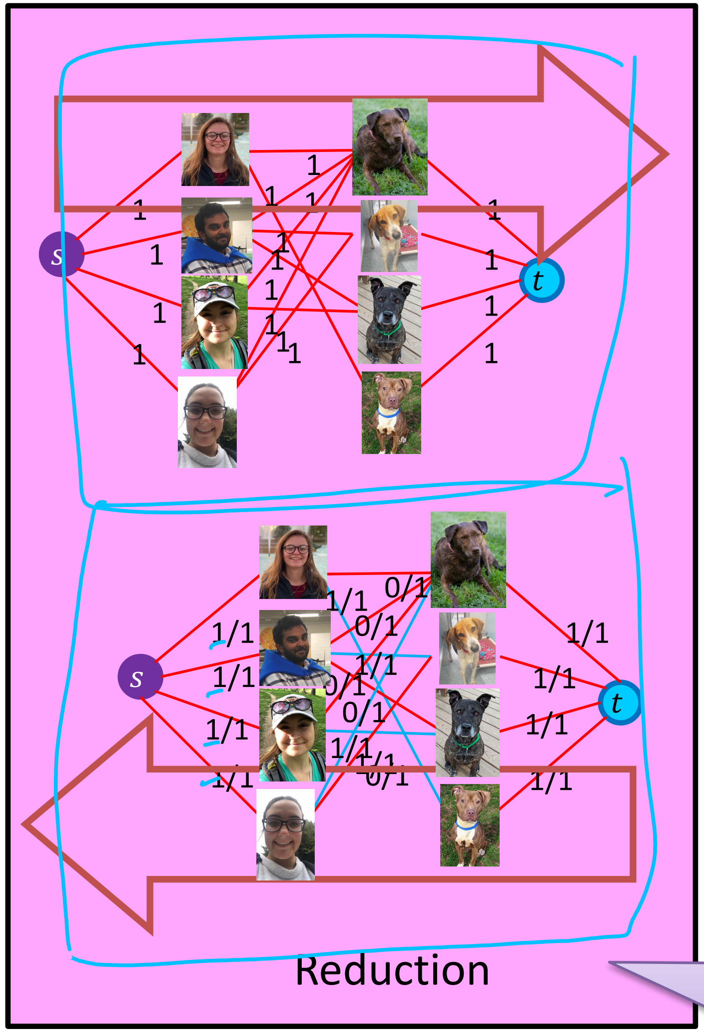


Bipartite Matching Reduction

Problem we don't know how to solve

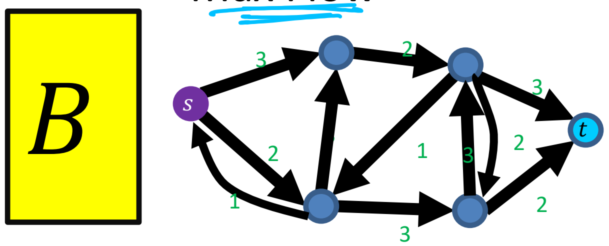
Bipartite Matching

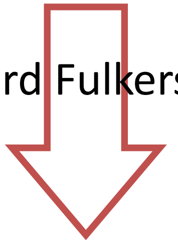
Solution for A



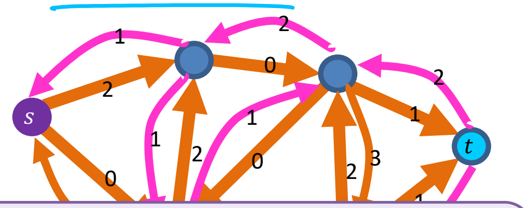
Problem we do know how to solve

Max Flow





Solution for B

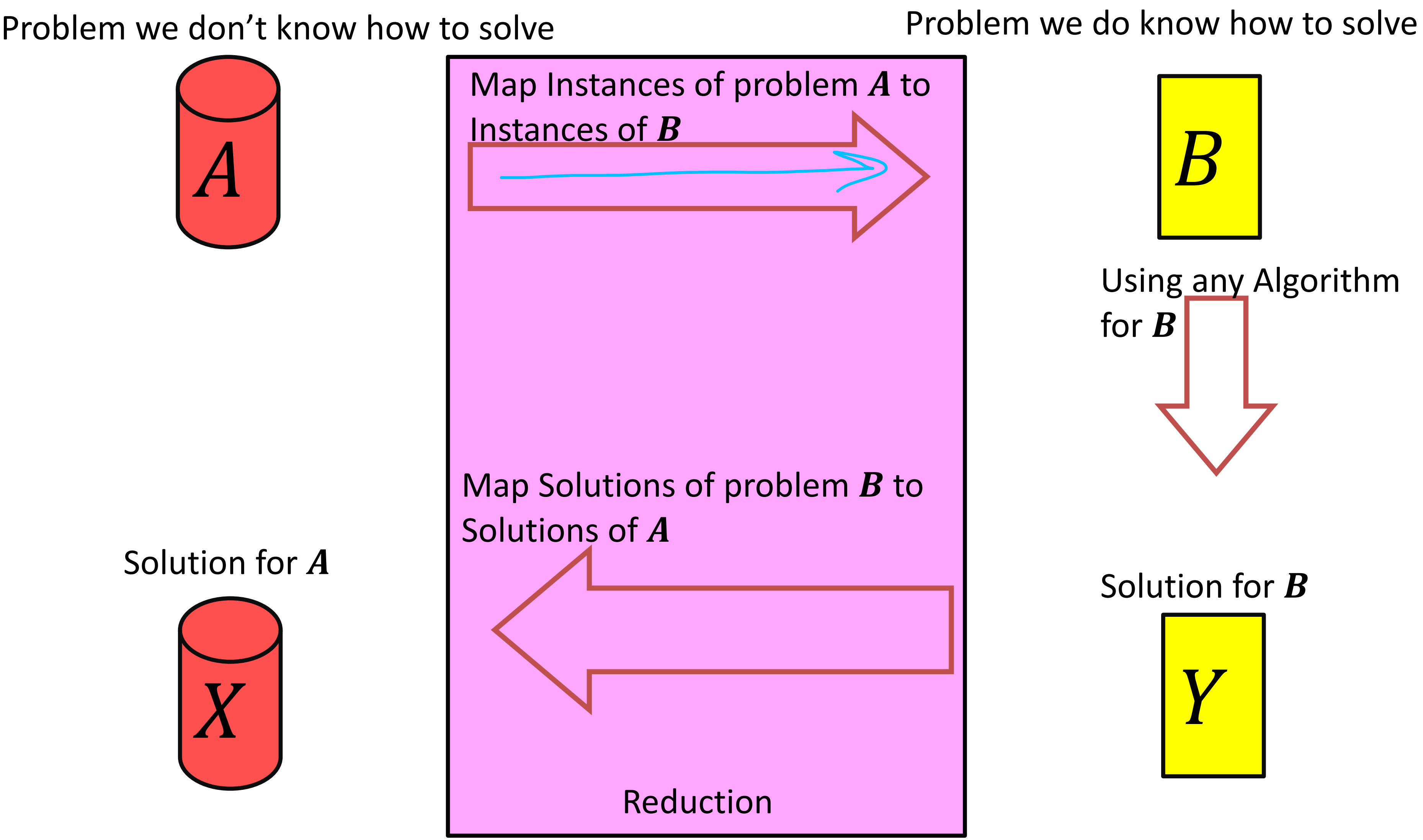


Must show (prove):
1) how to make construction
2) Why it works

prove 2 things
1) how to make constructions
2) why it works

- each person + each dog only participated in one matching
- max flow = max bipartite matching

In General: Reduction



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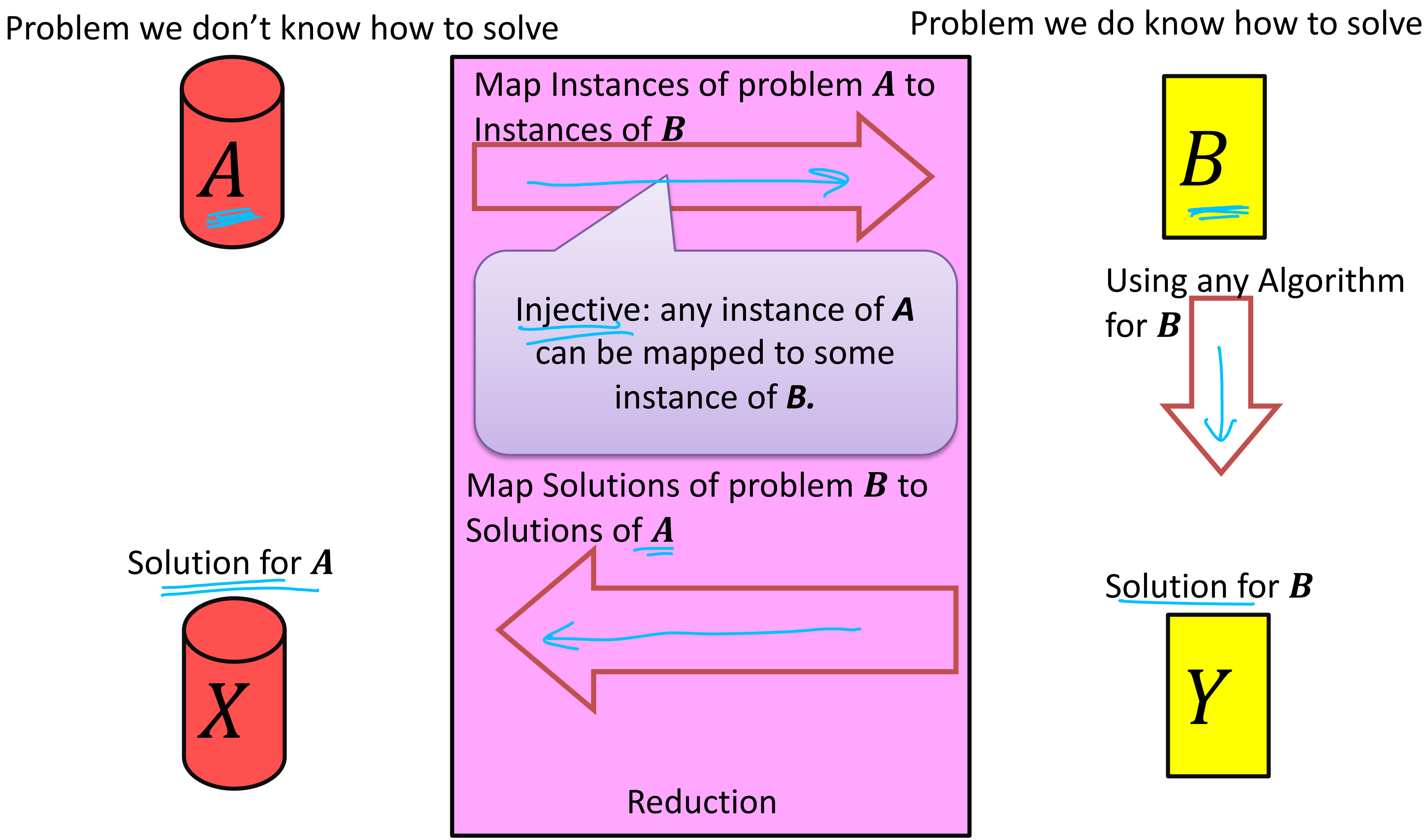
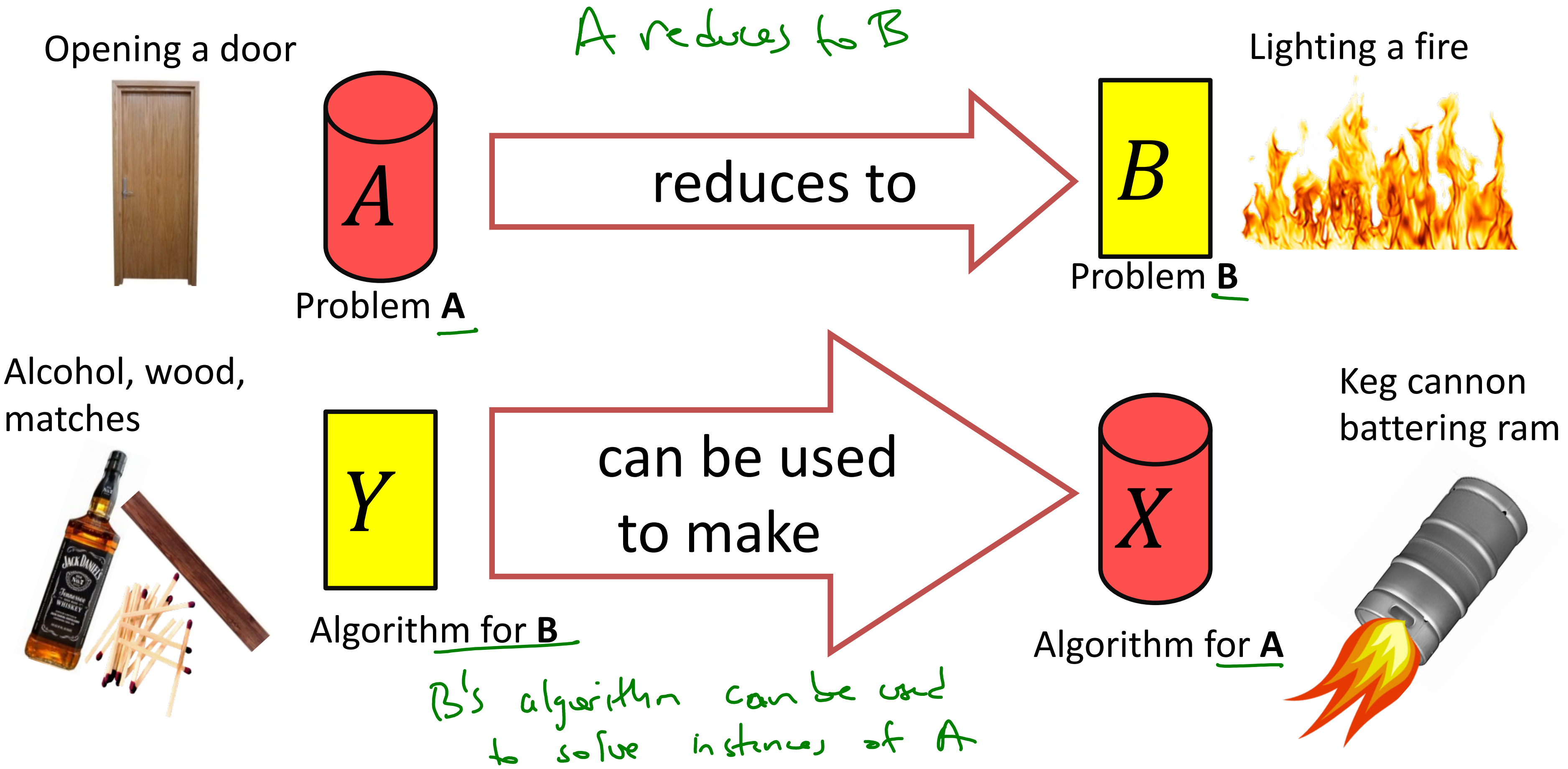


Worst-case lower-bound Proofs



Worst-case lower-bound Proofs

Opening a door

A

Problem **A**

reduces to

B

Problem **B**

Lighting a fire

Alcohol, wood, matches

Y

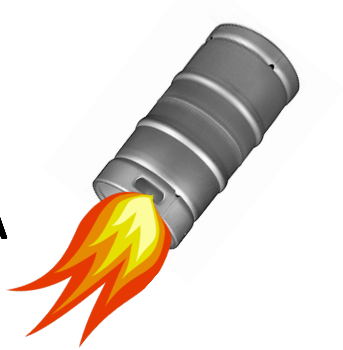
Algorithm for **B**

can be used to make

X

Algorithm for **A**

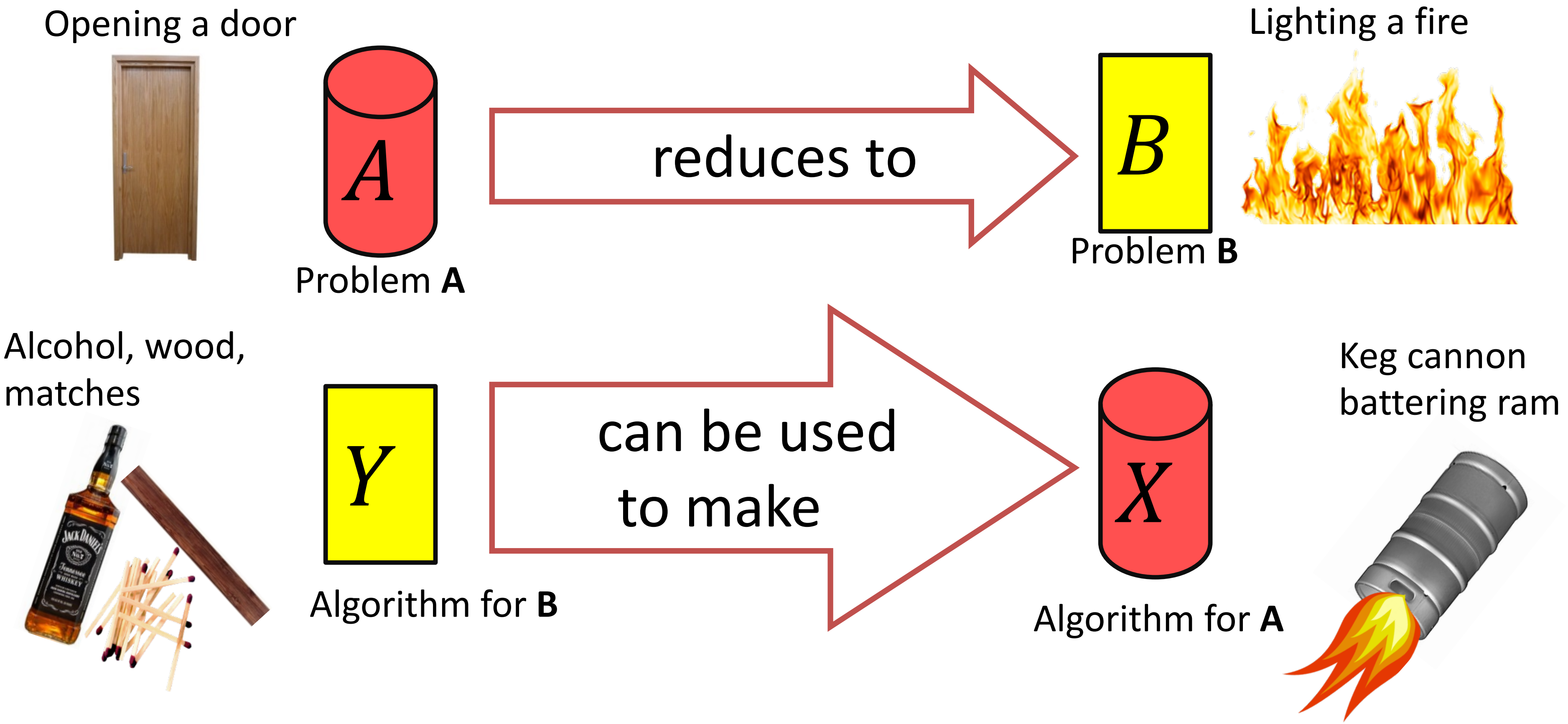
Keg cannon battering ram



A* is not a harder problem than *B

$\boldsymbol{A \leq B}$

Worst-case lower-bound Proofs



A is not a harder problem than B

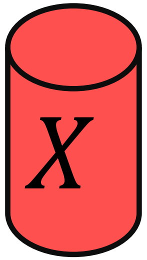
$$A \leq B$$

The name "reduces" is confusing: it is in the *opposite* direction of the making

Proof of Lower Bound by Reduction

<u>To Show: Y is slow</u>

Proof of Lower Bound by Reduction



<u>To Show: Y is slow</u>

1. We know X is slow (by a proof)
(e.g., X = some way to open the door)

Proof of Lower Bound by Reduction

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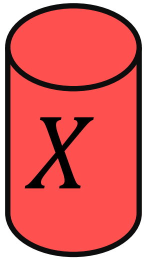

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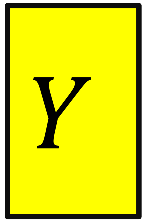
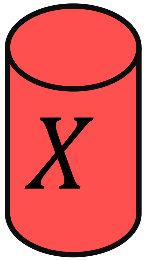


2. Assume Y is quick [toward contradiction]
(Y = some way to light a fire)

Proof of Lower Bound by Reduction

To Show: Y is slow



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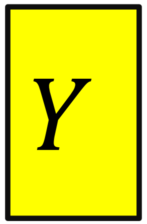

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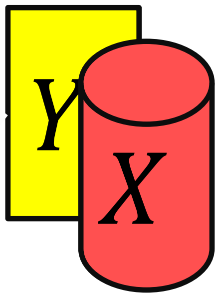
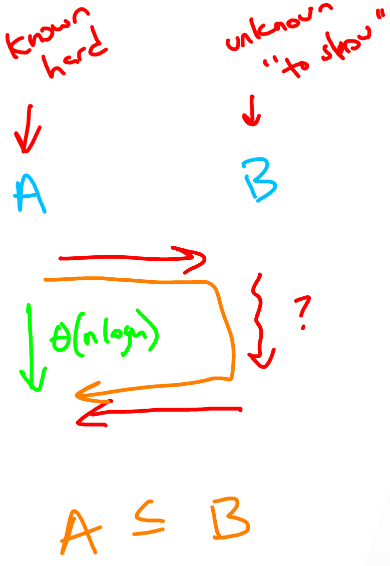


keg + air duct + fire

3. Show how to use Y to perform X quickly

Proof of Lower Bound by Reduction



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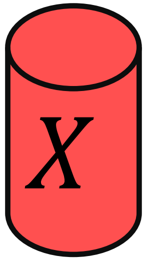
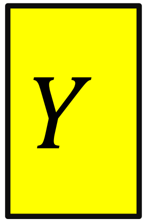


1. We know X is slow (by a proof)
(e.g., X = some way to open the door)

Sorting (comparison Based) $\Theta(n \log n)$



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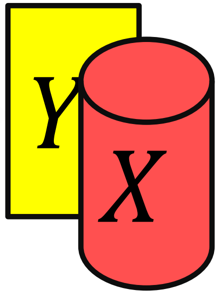

3. Show how to use Y to perform X quickly

- use Y to sort faster than $\Theta(n \log n)$

4. X is slow, but Y could be used to perform X quickly
conclusion: Y must not actually be quick