Warm up

Show that finding the minimum of an unordered list requires $\Omega(n)$ comparisons

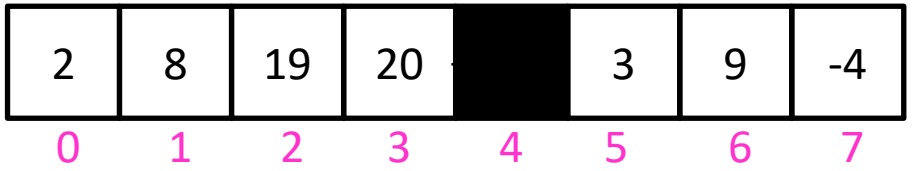
Find Min, Lower Bound Proof

Show that finding the minimum of an unordered list requires $\Omega(n)$ comparisons

Suppose (toward contradiction) that there is an algorithm for Find Min that does fewer than $\frac{n}{2} = \Omega(n)$ comparisons.

This means there is at least one "uncompared" element

We can't know that this element wasn't the min!



Homeworks

- HW4 due 11pm Thursday, February 27, 2020
  - Divide and Conquer and Sorting
  - Written (use LaTeX!)
  - Submit BOTH a pdf and a zip file (2 separate attachments)
- Midterm: March 4 (two weeks away!)
- Regrade Office Hours
  - Fridays 2:30pm-3:30pm (Rice 210)

Today's Keywords

- Sorting
- Linear time Sorting
- Counting Sort
- Radix Sort
- Maximum Sum Continuous Subarray

CLRS Readings

- Chapter 8

Sorting, so far

- Sorting algorithms we have discussed:
  - Mergesort $O(n \log n)$ Optimal!
  - Quicksort $O(n \log n)$ Optimal!
- Other sorting algorithms (will discuss):
  - Bubblesort $O(n^2)$
  - Insertionsort $O(n^2)$
  - Heapsort $O(n \log n)$ Optimal!

Speed Isn't Everything

Important properties of sorting algorithms:

- Run Time
  - Asymptotic Complexity
  - Constants
- In Place (or In-Situ)
  - Done with only constant additional space
- Adaptive
  - Faster if list is nearly sorted
- Stable
  - Equal elements remain in original order
- Parallelizable
  - Runs faster with multiple computers

Mergesort

- **Divide**:
  - Break n-element list into two lists of $n/2$ elements
- **Conquer**:
  - If $n > 1$: Sort each sublist recursively
  - If $n = 1$: List is already sorted (base case)
- **Combine**:
  - Merge together sorted sublists into one sorted list

Run Time?

$\Theta(n \log n)$

Optimal!

In Place?	Adaptive?	Stable?
No	No	Yes! (usually)

Merge

- **Combine**: Merge sorted sublists into one sorted list
- We have:
  - 2 sorted lists (L_1, L_2)
  - 1 output list (L_{out})

While (L_1 and L_2 not empty):

If $L_1[0] \leq L_2[0]$:

L_{out}.append(L_1.pop())

Else:

L_{out}.append(L_2.pop())

L_{out}.append(L_1)

L_{out}.append(L_2)

Stable:
If elements are equal, leftmost comes first

Mergesort

- **Divide**:
  - Break n-element list into two lists of $n/2$ elements
- **Conquer**:
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Optimal!

In Place?	Adaptive?	Stable?	Parallelizable?
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Mergesort

- **Divide:**
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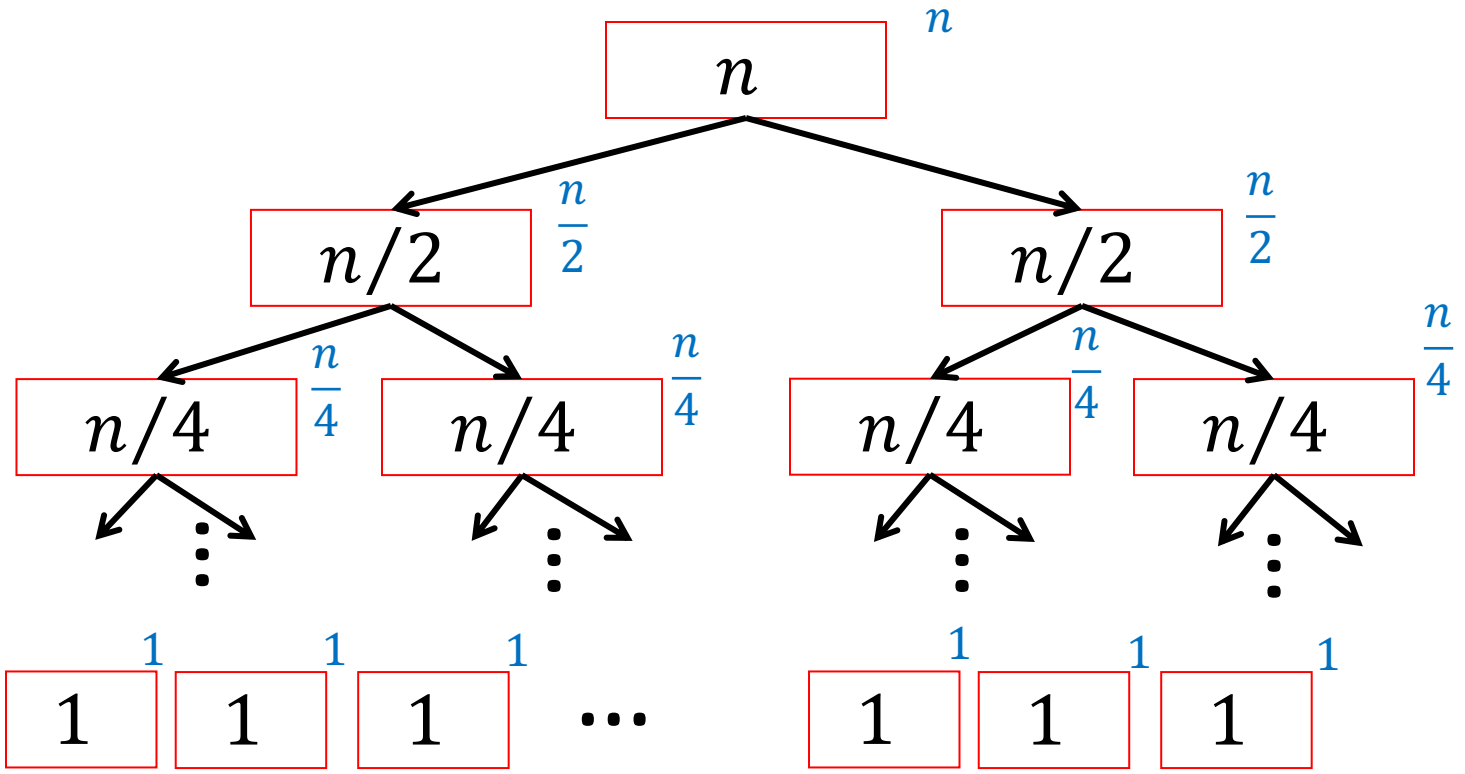
- **Conquer:**
  - If $n > 1$:
    - Sort each sublist recursively
  - If $n = 1$:
    - List is already sorted (base case)
- **Combine:**
  - Merge together sorted sublists into one sorted list

Parallelizable: Allow different machines to work on each sublist

Mergesort (Sequential)

$$T(n) = 2T(\frac{n}{2}) + n$$



n total / level

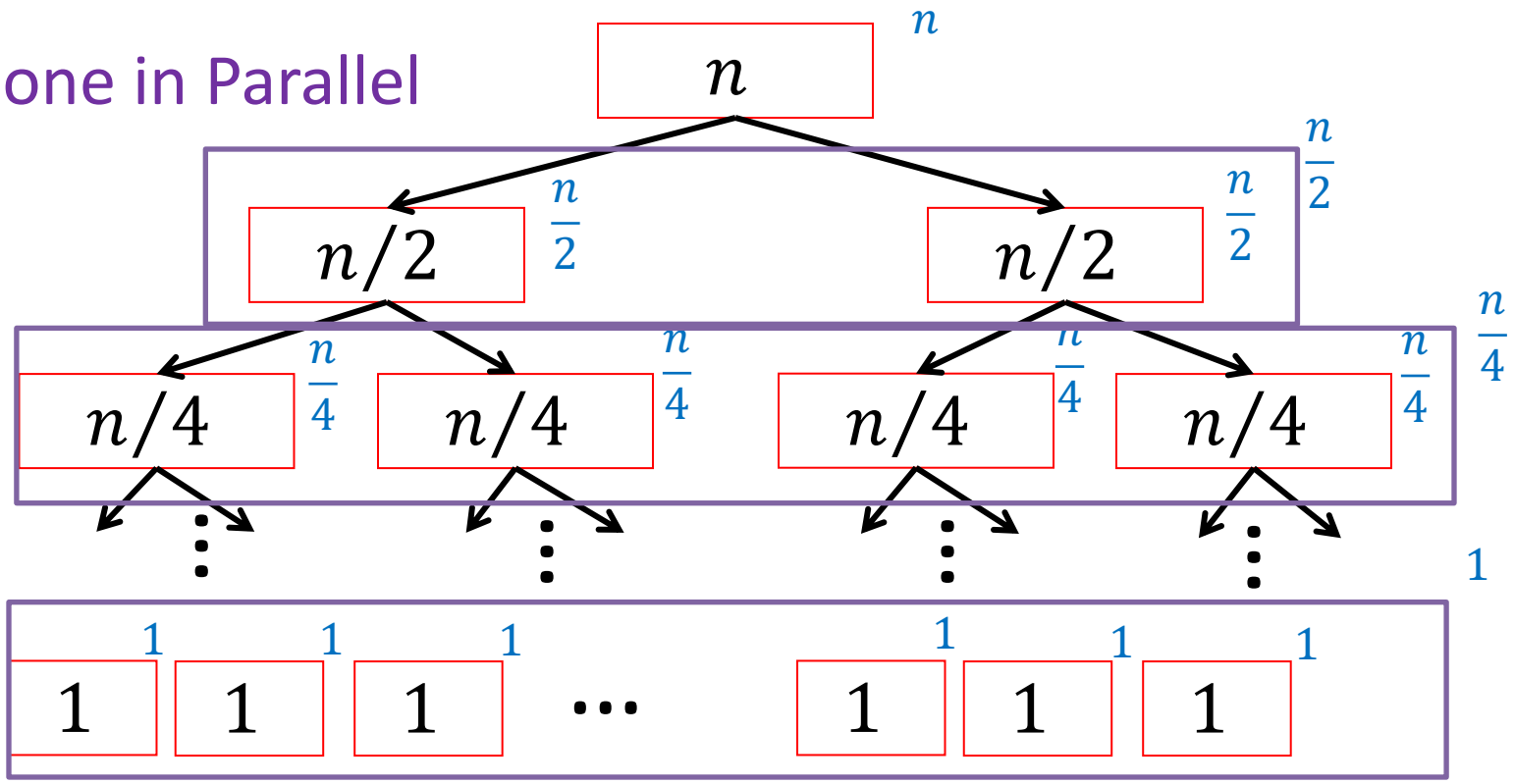
$\log_2 n$ levels of recursion

Run Time: $\Theta(n \log n)$

Mergesort (Parallel)

$$T(n) = T(\frac{n}{2}) + n$$

Done in Parallel



Run Time: $\Theta(n)$

Quicksort

Idea: pick a partition element, recursively sort two sublists around that element

- Divide: select an element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Run Time?

$\Theta(n \log n)$

(almost always)

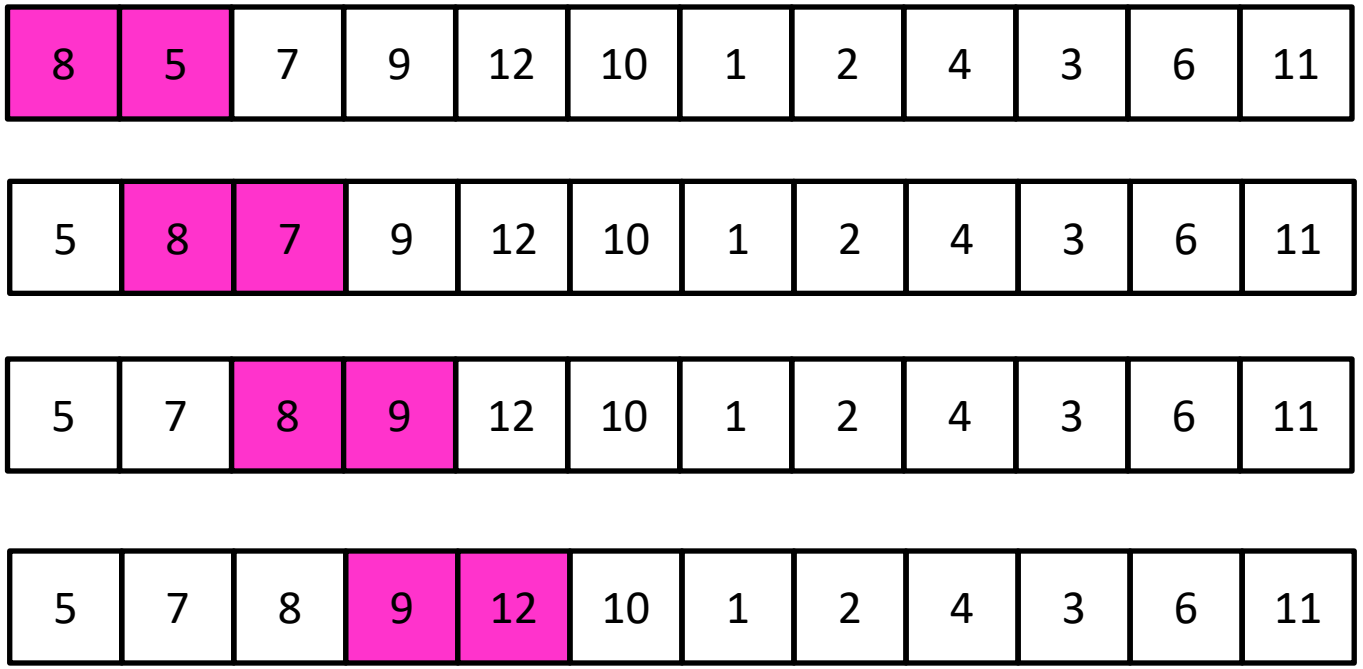
Better constants than Mergesort

In Place?	Adaptive?	Stable?	Parallelizable?
kinda	No!	No	Yes!

Uses stack for recursive calls

Bubble Sort

Idea: March through list, swapping adjacent elements if out of order, repeat until sorted



Bubble Sort

- Idea: March through list, swapping adjacent elements if out of order, repeat until sorted

Run Time?

$\Theta(n^2)$

Constants worse than Insertion Sort

In Place?

Yes

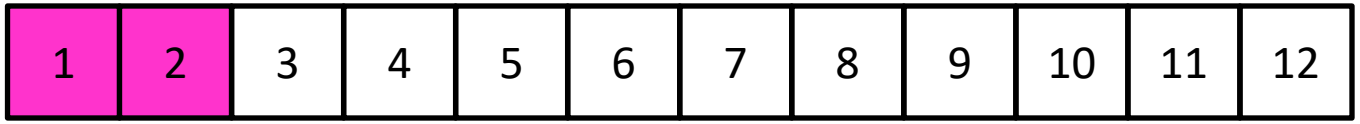
Adaptive?

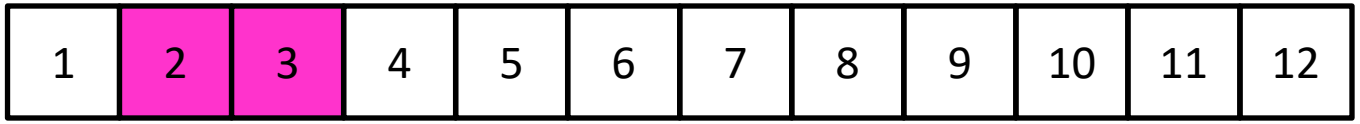
Kinda

"Compared to straight insertion […], bubble sorting requires a more complicated program and takes about twice as long!" –Donald Knuth

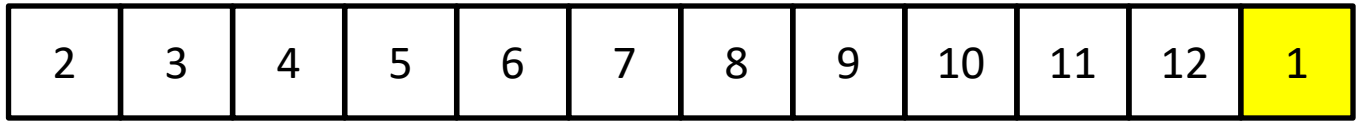
Bubble Sort is "almost" Adaptive

Idea: March through list, swapping adjacent elements if out of order

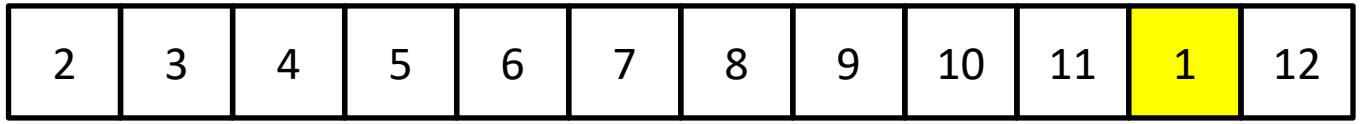




Only makes one "pass"



After one "pass"



Requires n passes, thus is $O(n^2)$

Bubble Sort

- Idea: March through list, swapping adjacent elements if out of order, repeat until sorted

Run Time?

$\Theta(n^2)$

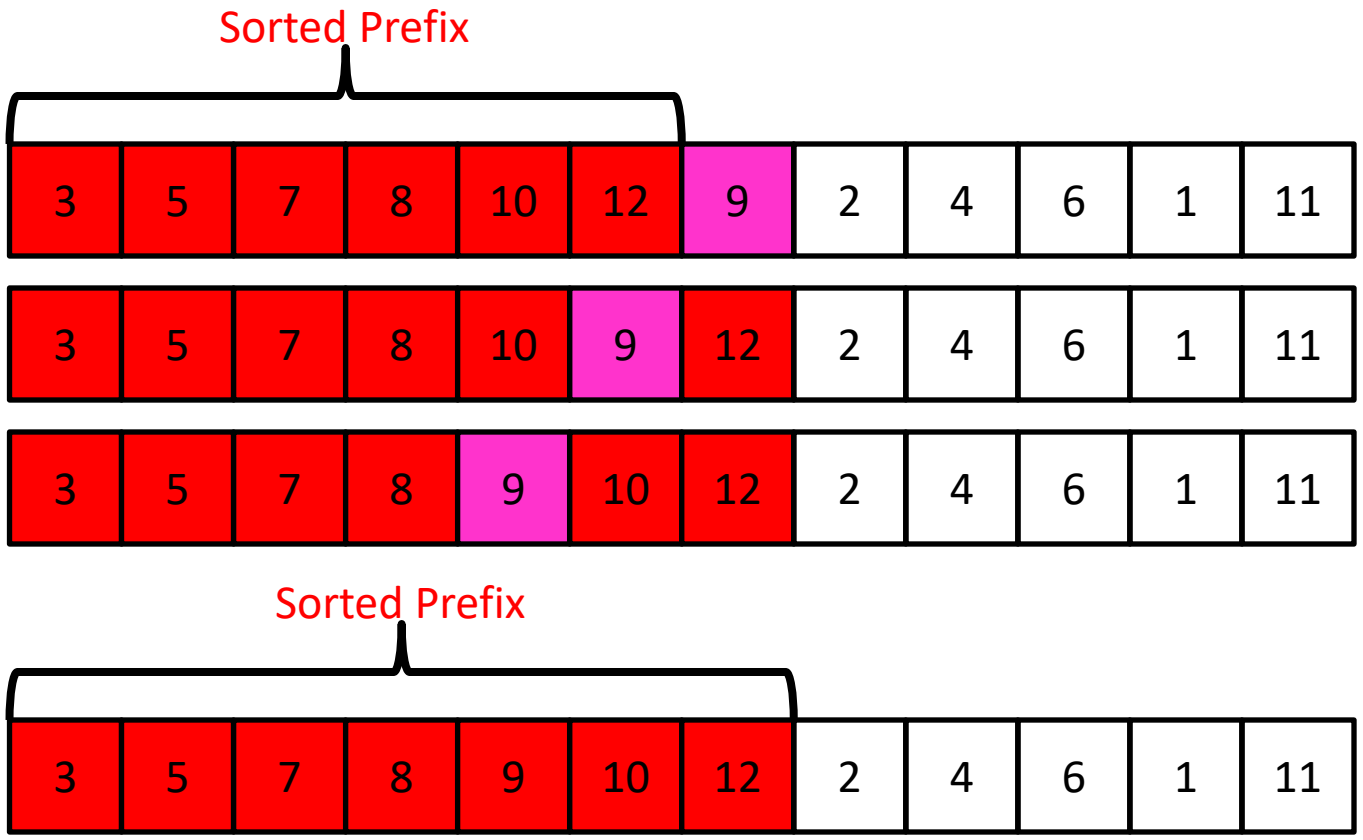
Constants worse than Insertion Sort

In Place?	Adaptive?	Stable?	Parallelizable?
Yes!	~~Kinda~~ Not really	Yes	No

"the bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems" –Donald Knuth, The Art of Computer Programming

Insertion Sort

Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element

Run Time?

$\Theta(n^2)$

(but with very small constants)

Great for short lists!

In Place?

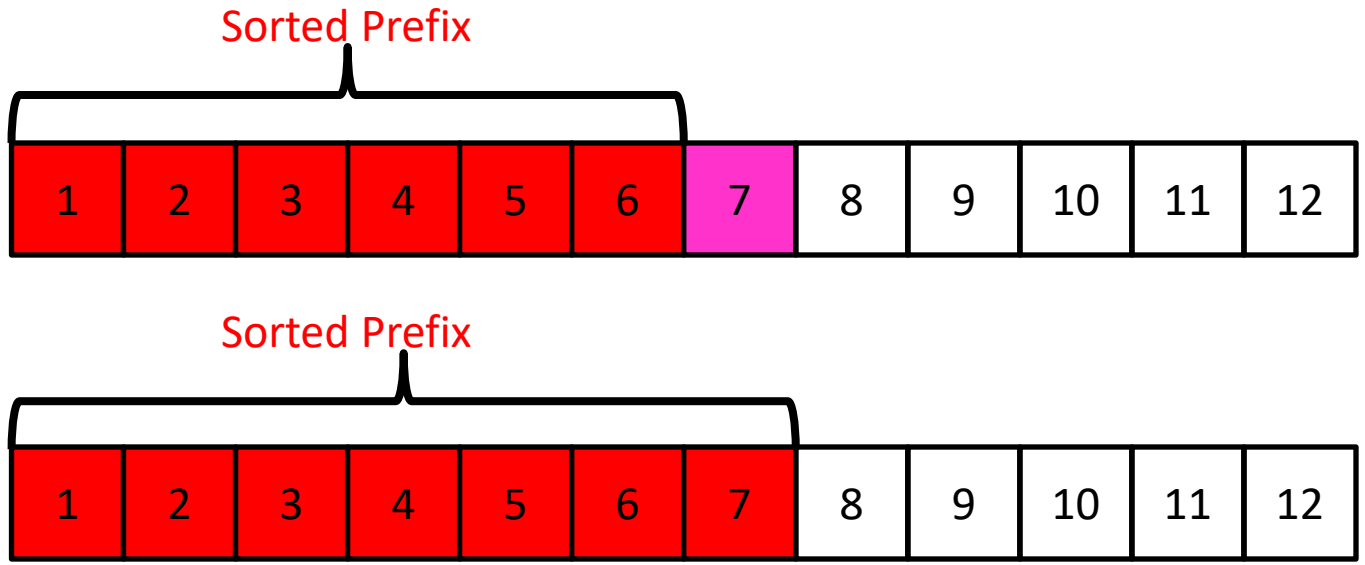
Yes!

Adaptive?

Yes

Insertion Sort is Adaptive

Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Only one comparison needed per element! Runtime: $O(n)$

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element

Run Time?

$\Theta(n^2)$

(but with very small constants)

Great for short lists!

In Place?

Yes!

Adaptive?

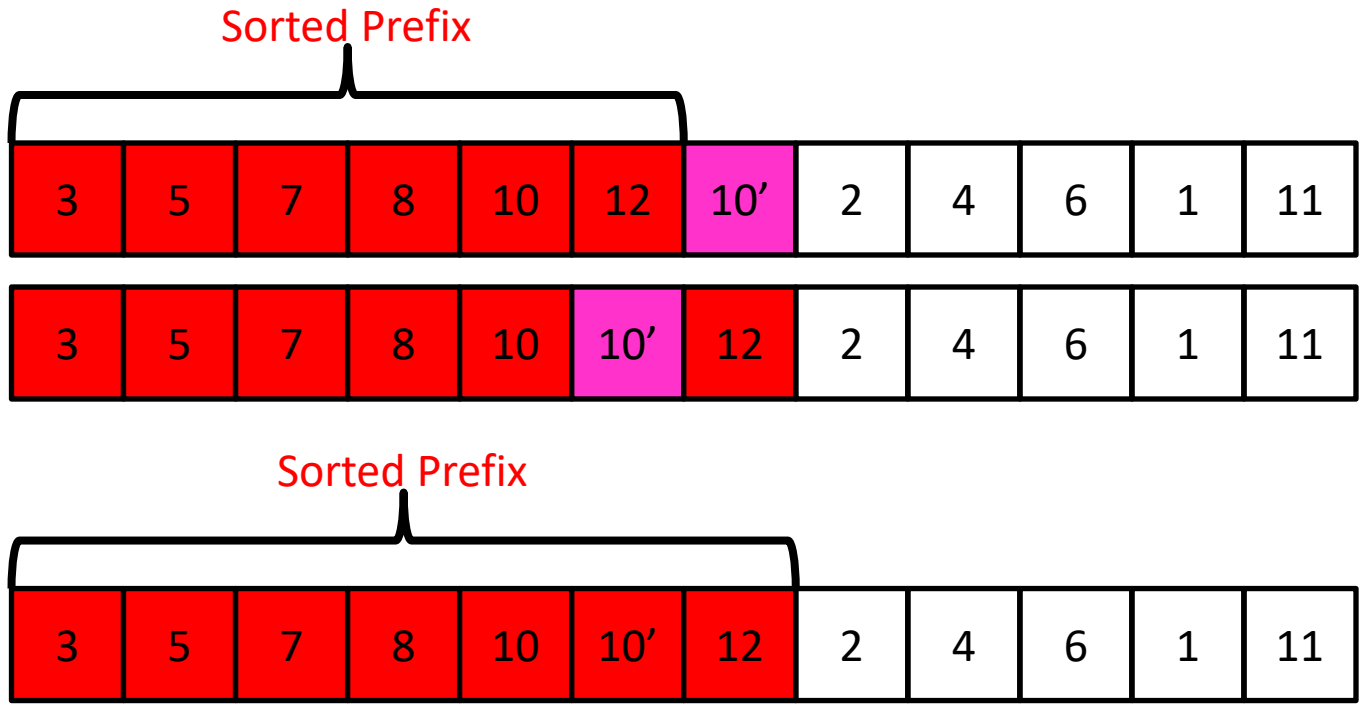
Yes

Stable?

Yes

Insertion Sort is Stable

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



The "second" 10 will stay to the right

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element

Run Time?

$\Theta(n^2)$

(but with very small constants)

Great for short lists!

In Place?	Adaptive?	Stable?	Parallelizable?
Yes!	Yes	Yes	No

"All things considered, it's actually a pretty good sorting algorithm!" –Nate Brunelle

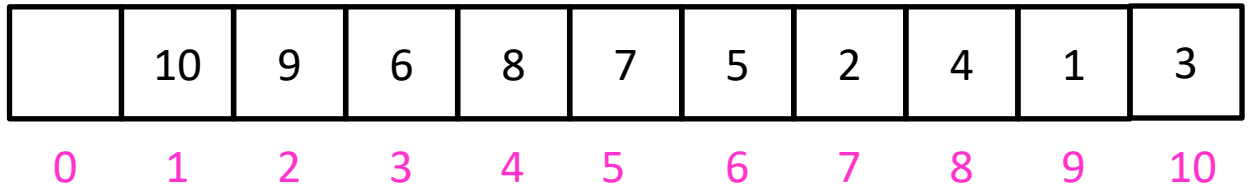
Can sort a list as it is received, i.e., don't need the entire list to begin sorting

Online?

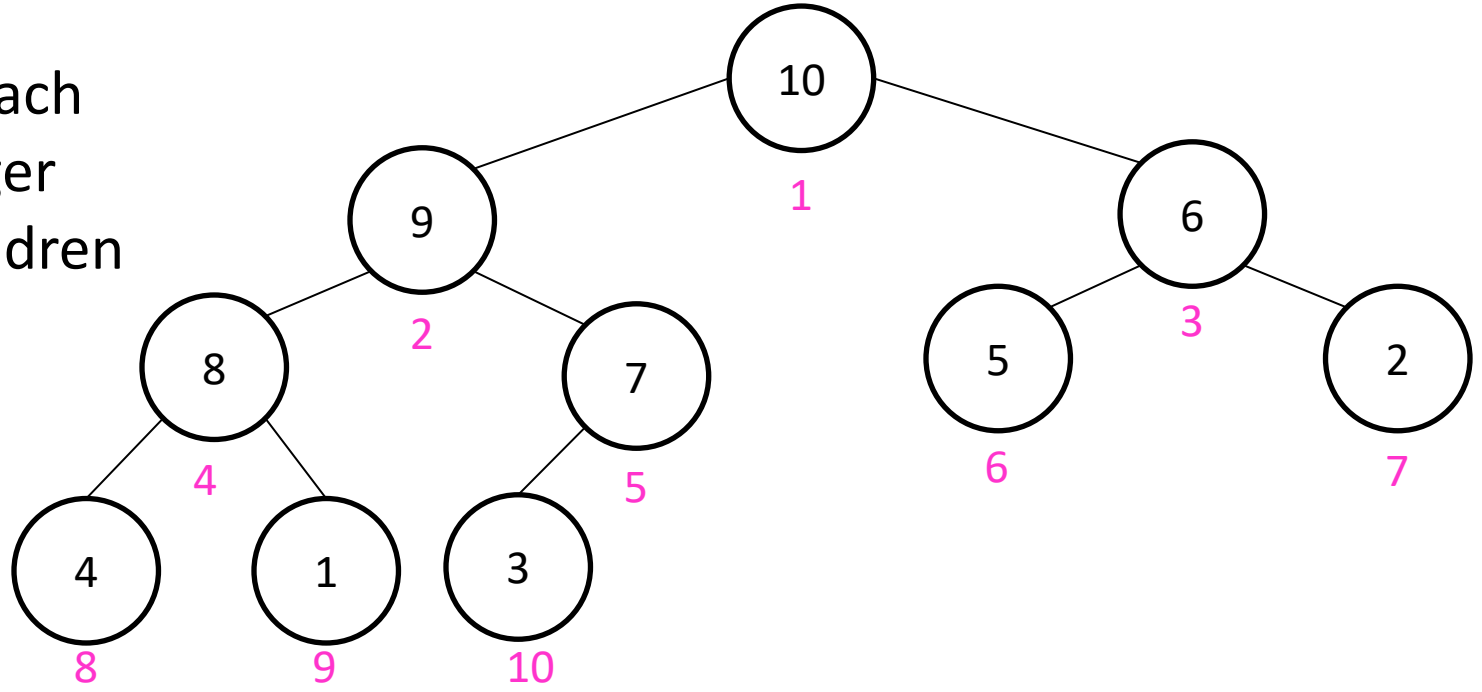
Yes

Heap Sort

- Idea: Build a Heap, repeatedly extract max element from the heap to build sorted list Right-to-Left

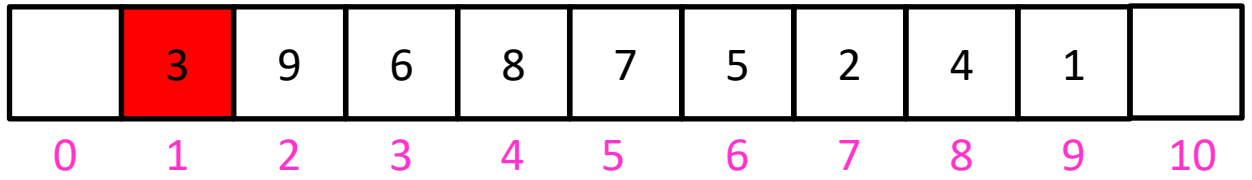


Max Heap Property: Each node is larger than its children



Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call Heapify(root)



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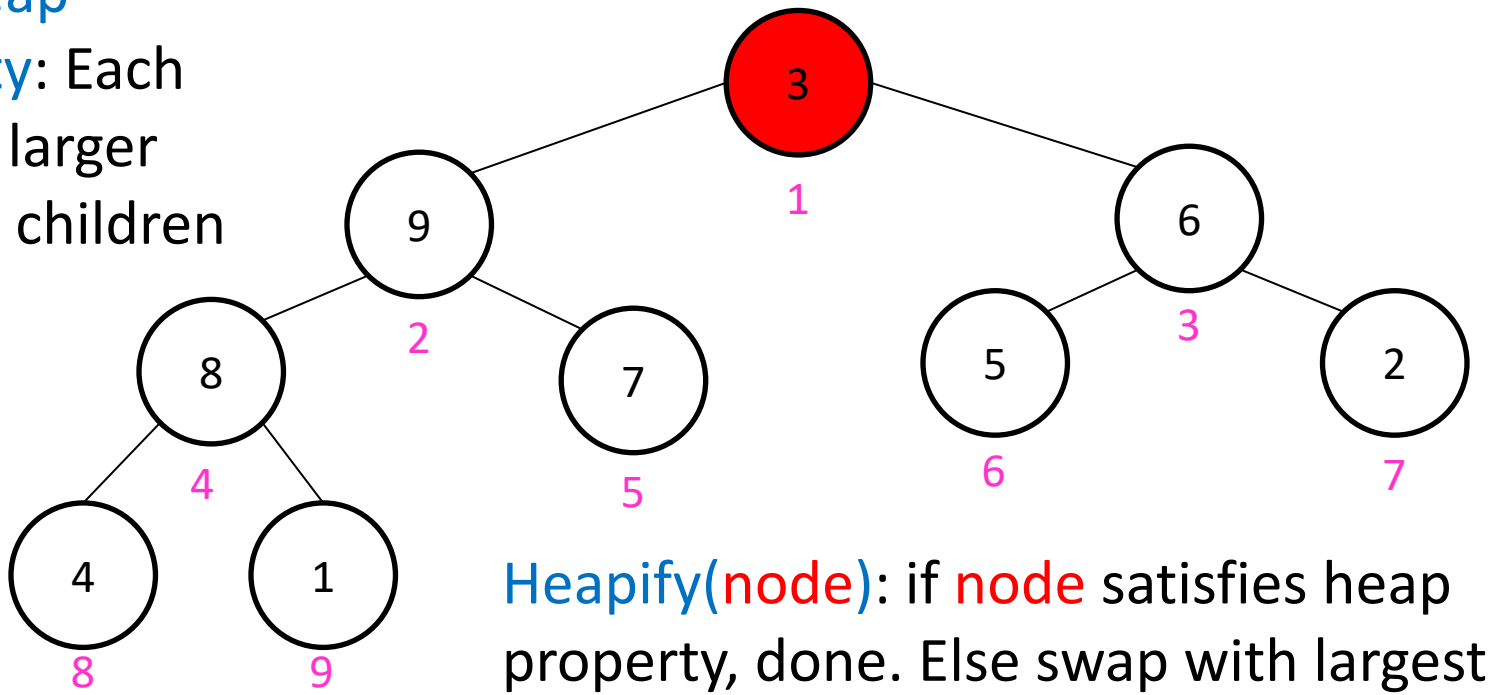

Heapify(node): if node satisfies heap property, done. Else swap with largest child and recurse on that subtree

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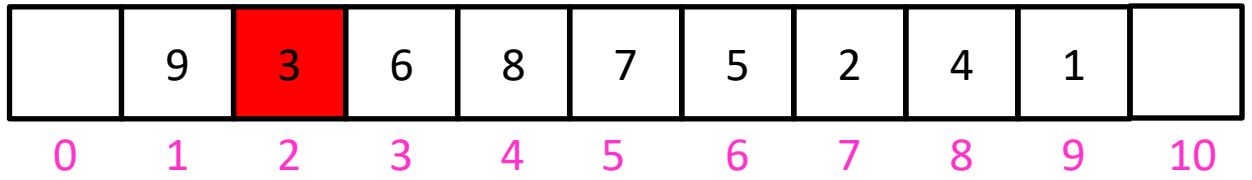

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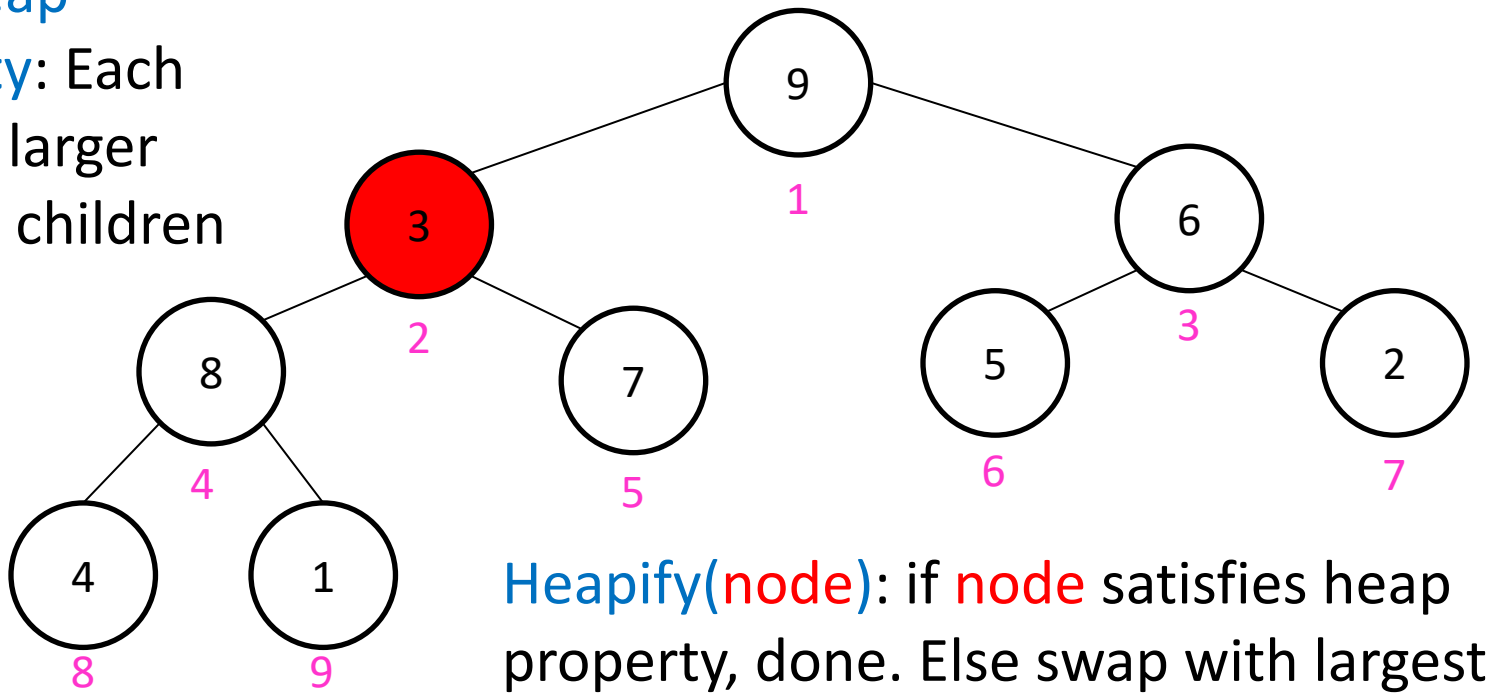

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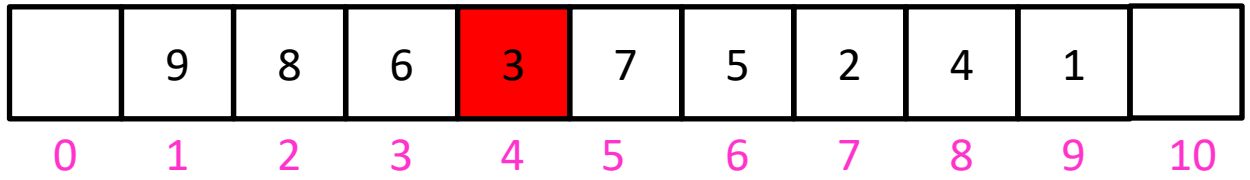

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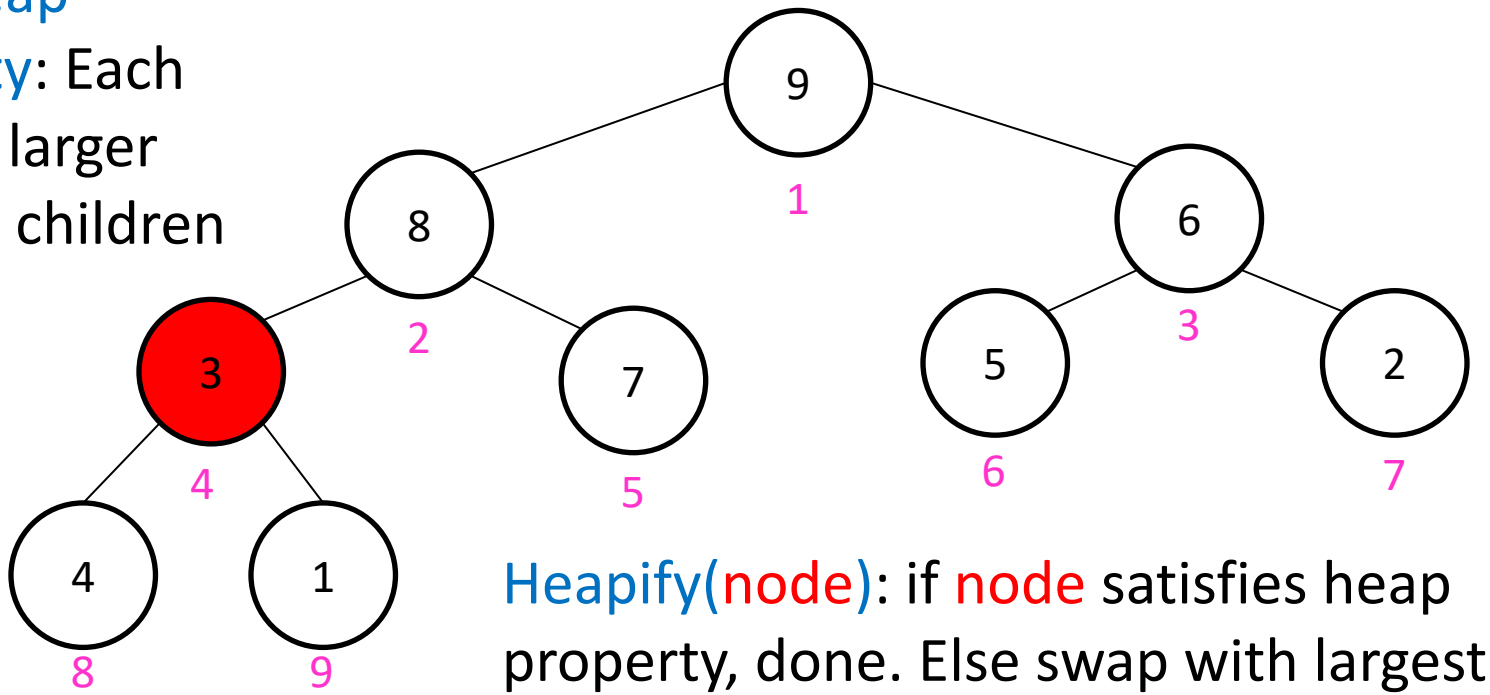

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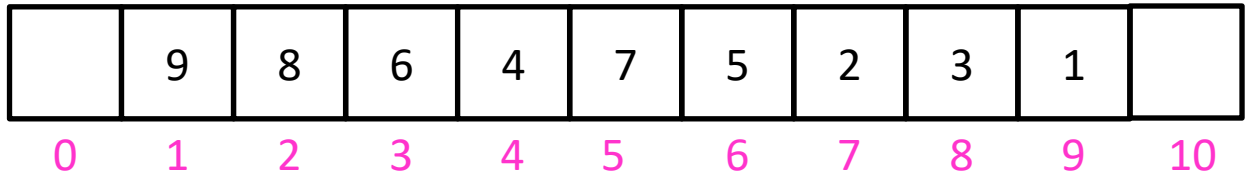

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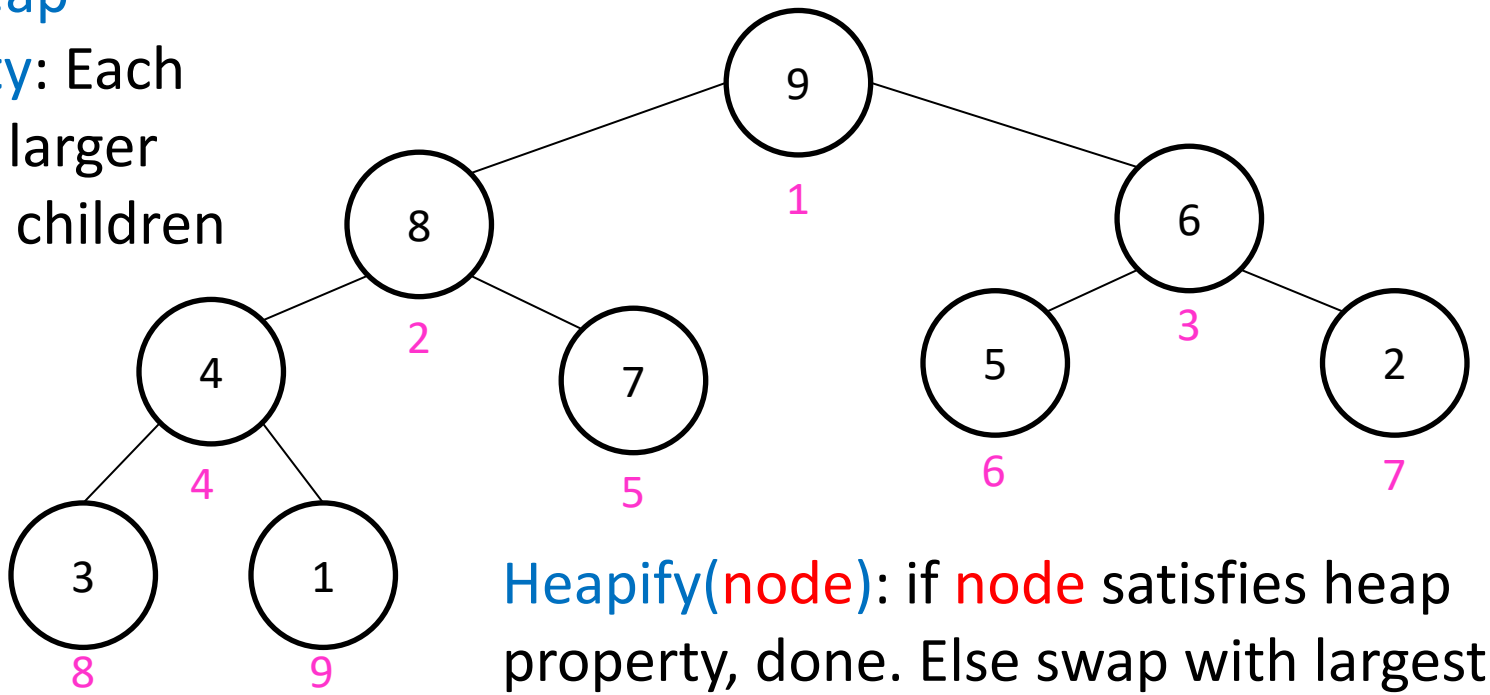

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Heap Sort

- Idea: Build a Heap, repeatedly extract max element from the heap to build sorted list Right-to-Left

Run Time?

$\Theta(n \log n)$

Constants worse than Quick Sort

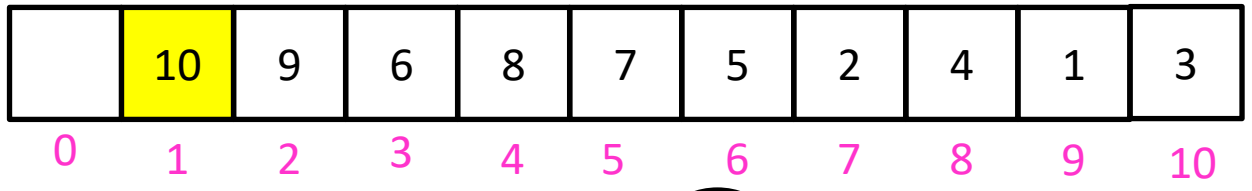
In Place?

Yes!

When removing an element from the heap, move it to the (now unoccupied) end of the list

In Place Heap Sort

- Idea: When removing an element from the heap, move it to the (now unoccupied) end of the list



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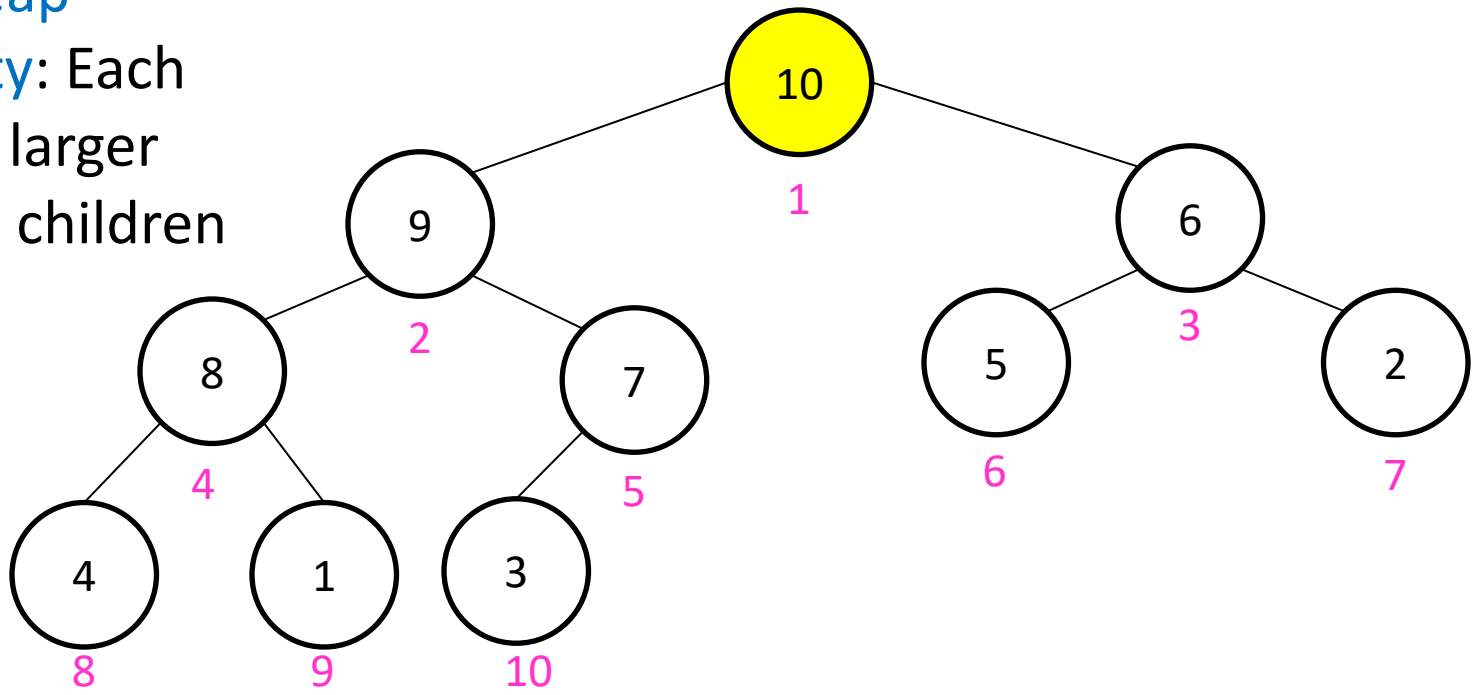

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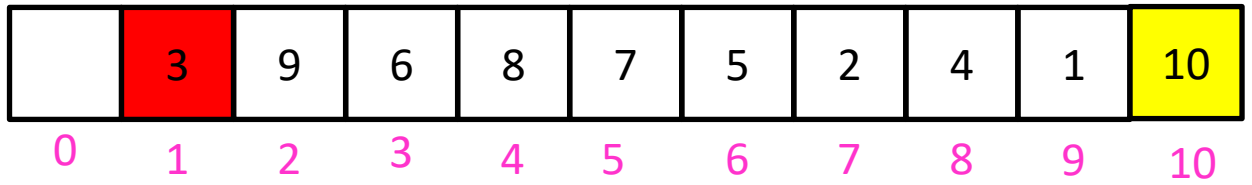

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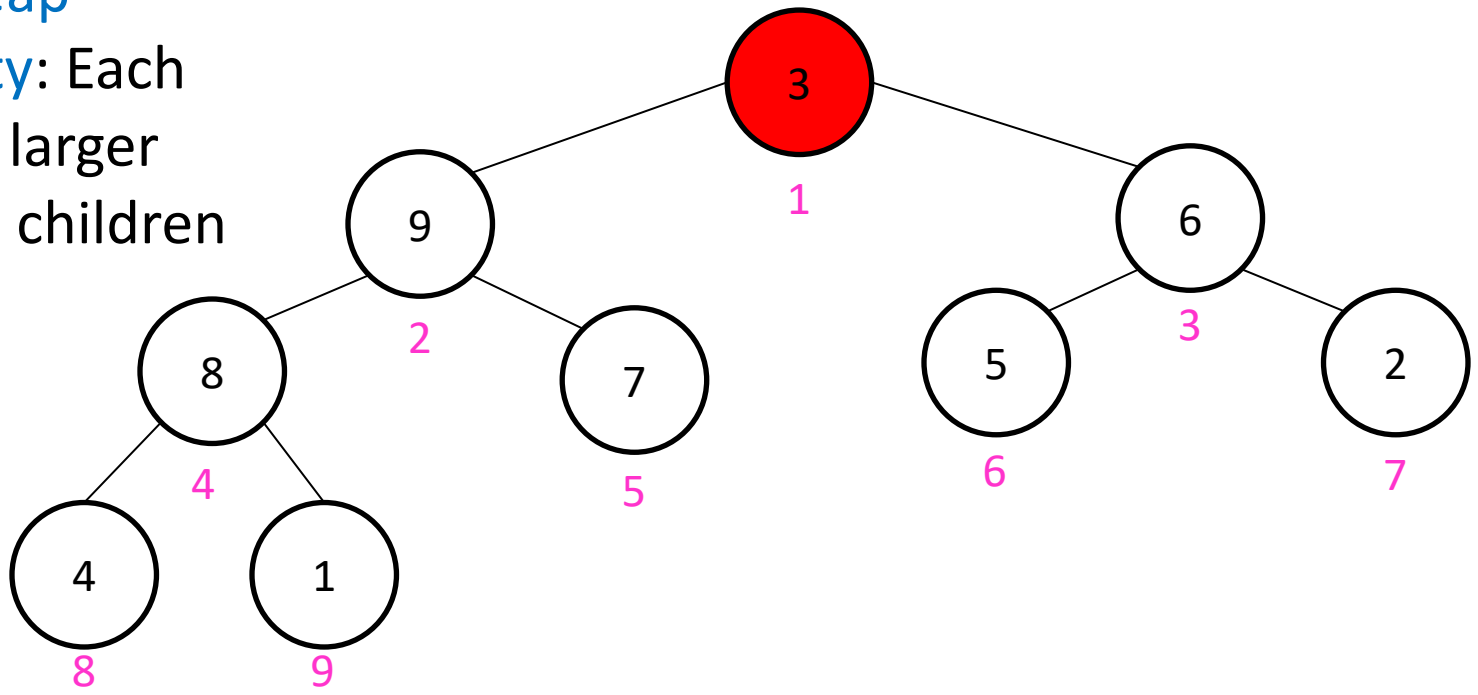

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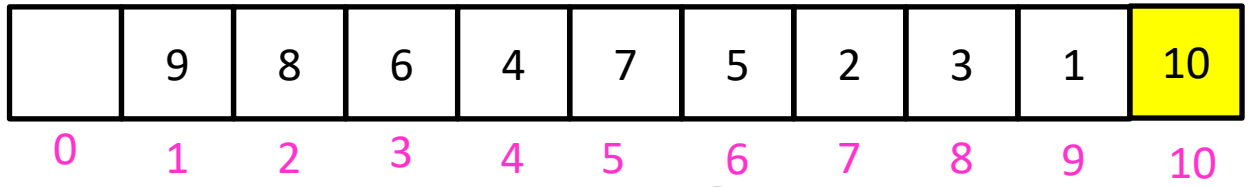

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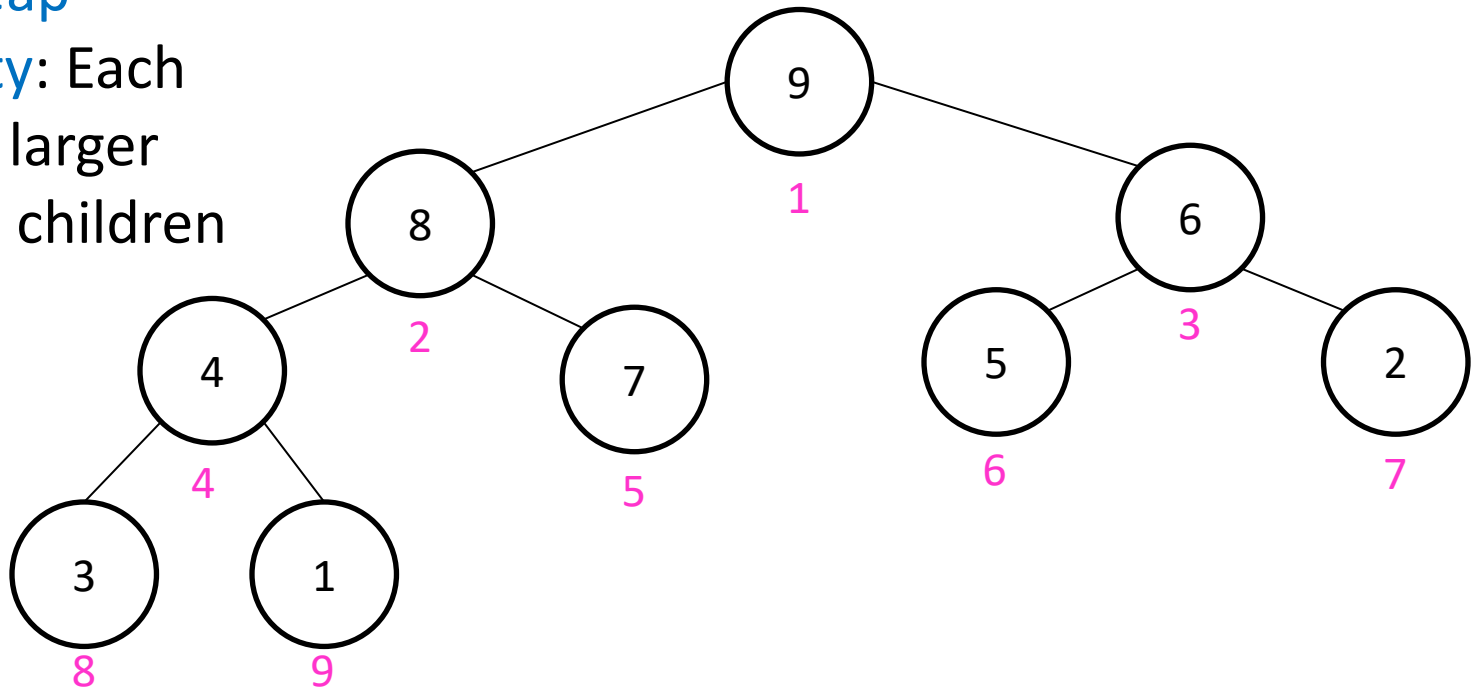

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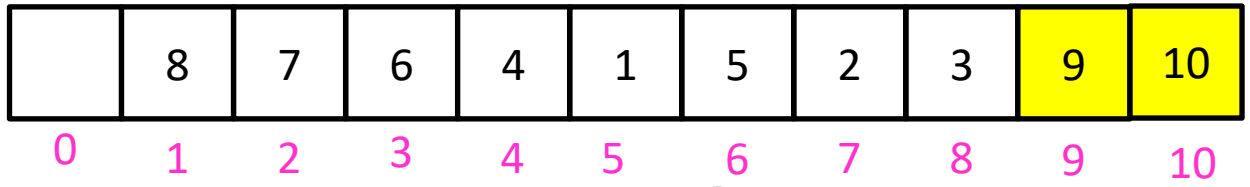

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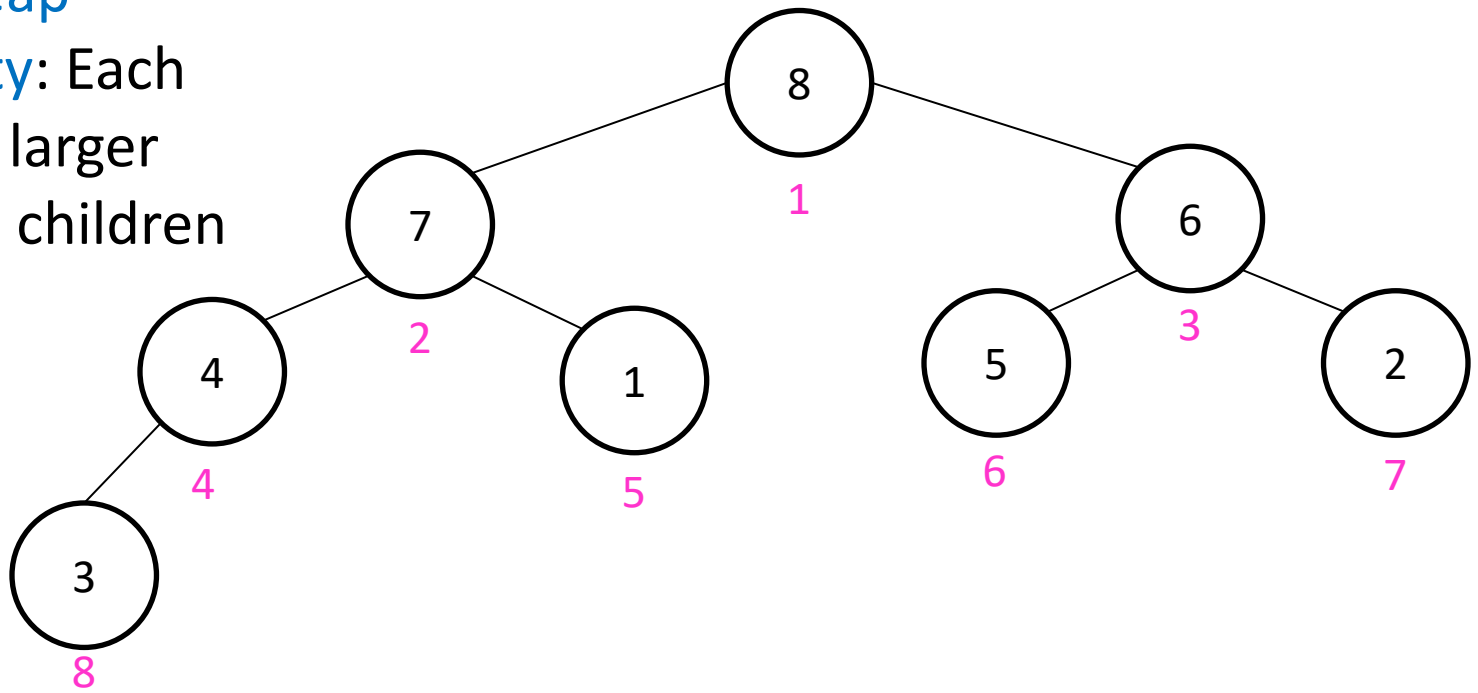

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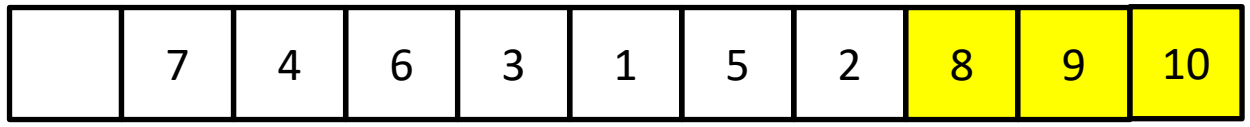

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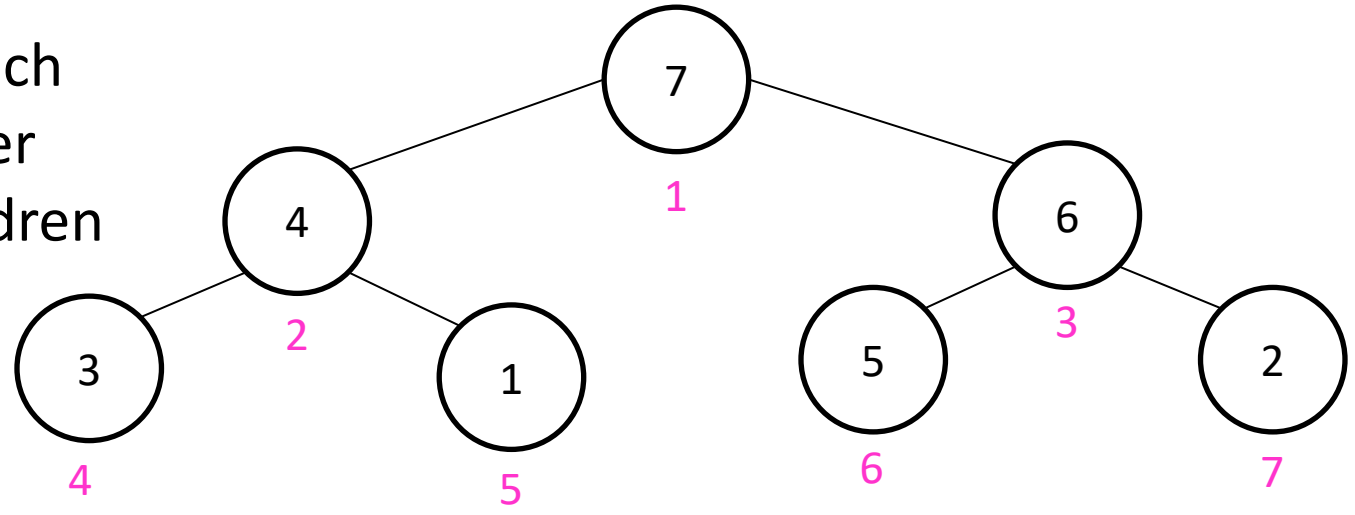


Heap Sort

- Idea: Build a Heap, repeatedly extract max element from the heap to build sorted list Right-to-Left

Run Time?

$\Theta(n \log n)$

Constants worse than Quick Sort

In Place?	Adaptive?	Stable?	Parallelizable?
Yes!	No	HW4 EC	No

Sorting, so far

- Sorting algorithms we have discussed:
  - Mergesort $O(n \log n)$ Optimal!
  - Quicksort $O(n \log n)$ Optimal!
- Other sorting algorithms (will discuss):
  - Bubblesort $O(n^2)$
  - Insertionsort $O(n^2)$
  - Heapsort $O(n \log n)$ Optimal!

Sorting in Linear Time

- Cannot be comparison-based
- Need to make some sort of assumption about the contents of the list
  - Small number of unique values
  - Small range of values
  - Etc.

Counting Sort

- Idea: Count how many things are less than each element

$L =$

3	6	6	1	3	4	1	6
1	2	3	4	5	6	7	8

1. Range is $[1, k]$ (here $[1,6]$)
   make an array C of size k
   populate with counts of each value

For i in L:
$$++C[L[i]]$$

$C =$

2	0	2	1	0	3
1	2	3	4	5	6

running sum

2. Take "running sum" of C
   to count things less than each value

For $i = 1$ to $\text{len}(C)$:
$$C[i] = C[i-1] + C[i]$$

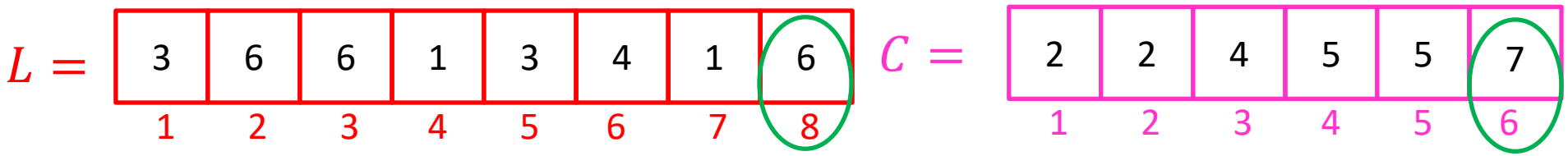
$C =$

2	2	4	5	5	8
1	2	3	4	5	6

To sort: last item of value 3 goes at index 4

Counting Sort

- Idea: Count how many things are less than each element

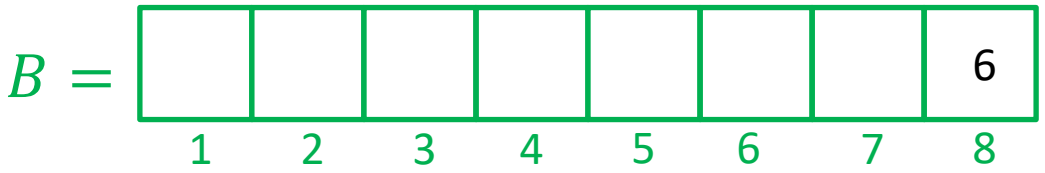


For each element of L (last to first):
Use C to find its proper place in B
Decrement that position of C

For $i = \text{len}(L)$ downto 1:

$$B\left[C\left[L[i]\right]\right] = L[i]$$

$$C\left[L[i]\right] = C\left[L[i]\right] - 1$$



Counting Sort

- Idea: Count how many things are less than each element

$L =$

3	6	6	1	3	4	1	6
1	2	3	4	5	6	7	8

$C =$

1	2	4	5	5	7
1	2	3	4	5	6

Last item of value 1 goes at index 2

For each element of L (last to first):
Use C to find its proper place in B
Decrement that position of C

For $i = \text{len}(L)$ downto 1:

$$B\left[C\left[L[i]\right]\right] = L[i]$$

$$C\left[L[i]\right] = C\left[L[i]\right] - 1$$

$B =$

	1						6
1	2	3	4	5	6	7	8

Run Time: $O(n + k)$

Memory: $O(n + k)$

Counting Sort

- Why not always use counting sort?
- For 64-bit numbers, requires an array of length $2^{64} > 10^{19}$
  - 5 GHz CPU will require > 116 years to initialize the array
  - 18 Exabytes of data
    - Total amount of data that Google has (?)

One Exabyte = 10^{18} bytes
1 million terabytes (TB)
1 billion gigabytes (GB)

100,000 x Library of Congress (print)

12 Exabytes

https://en.wikipedia.org/wiki/Utah_Data_Center

Radix Sort

- Idea: Stable sort on each digit, from least significant to most significant

103	801	401	323	255	823	999	101	113	901	555	512	245	800	018	121
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

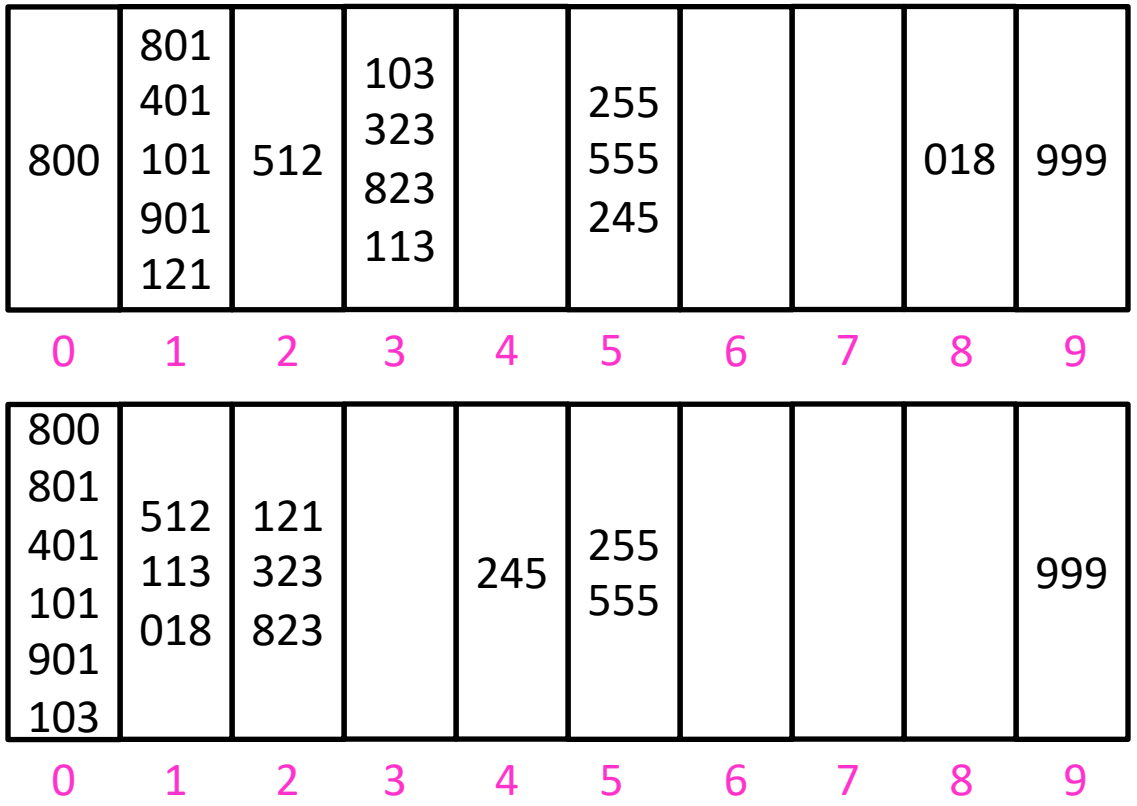
Place each element into a "bucket" according to its 1's place

0	1	2	3	4	5	6	7	8	9
800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999

Radix Sort

- Idea: Stable sort on each digit, from least significant to most significant

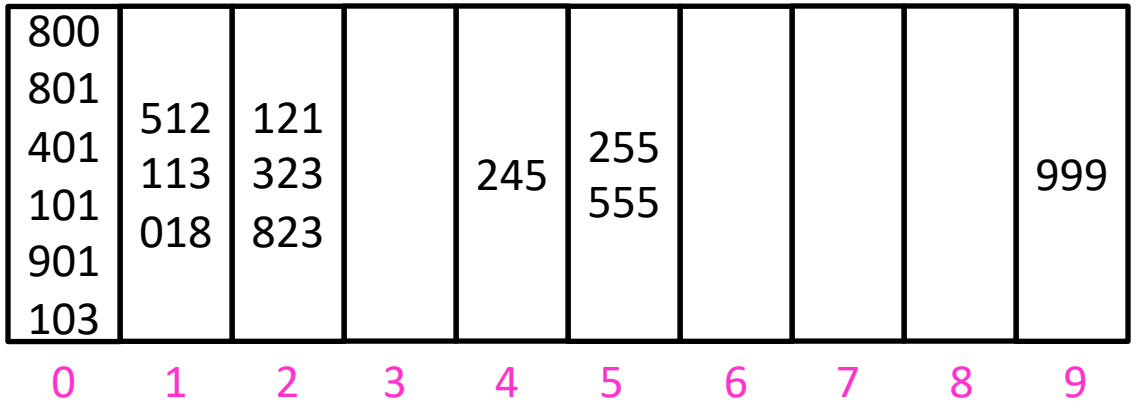
Place each element into a “bucket” according to its 10’s place

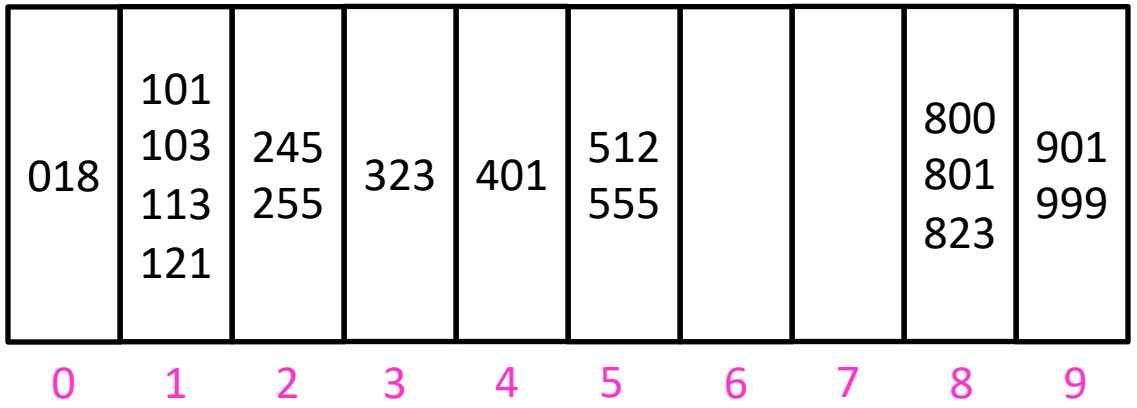


Radix Sort

- Idea: Stable sort on each digit, from least significant to most significant

Place each element into a "bucket" according to its 100's place





Run Time: $O\big(d(n+b)\big)$

d = digits in largest value

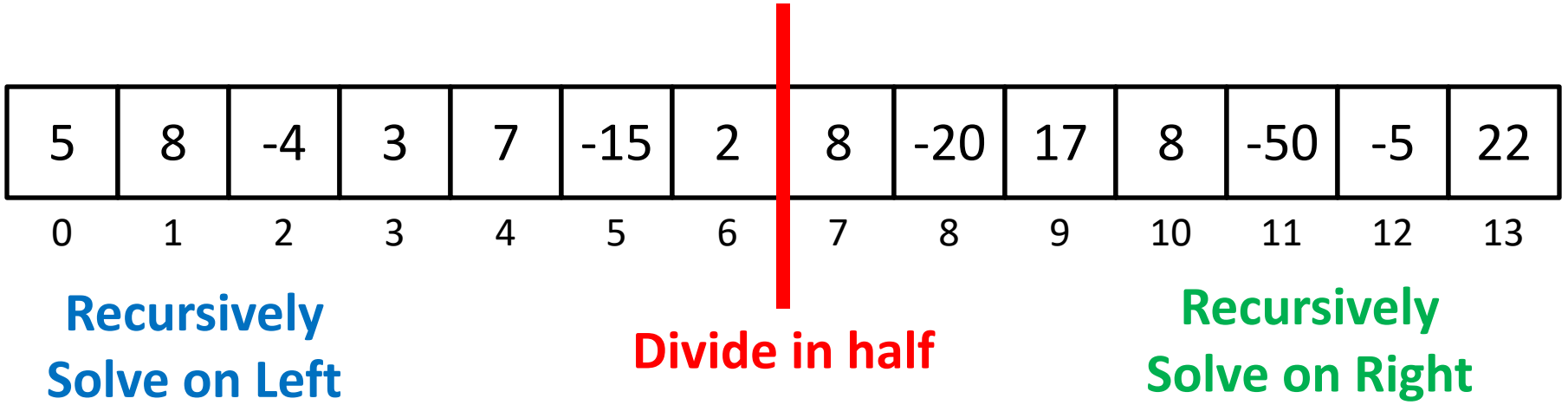
b = base of representation

Maximum Sum Continuous Subarray Problem

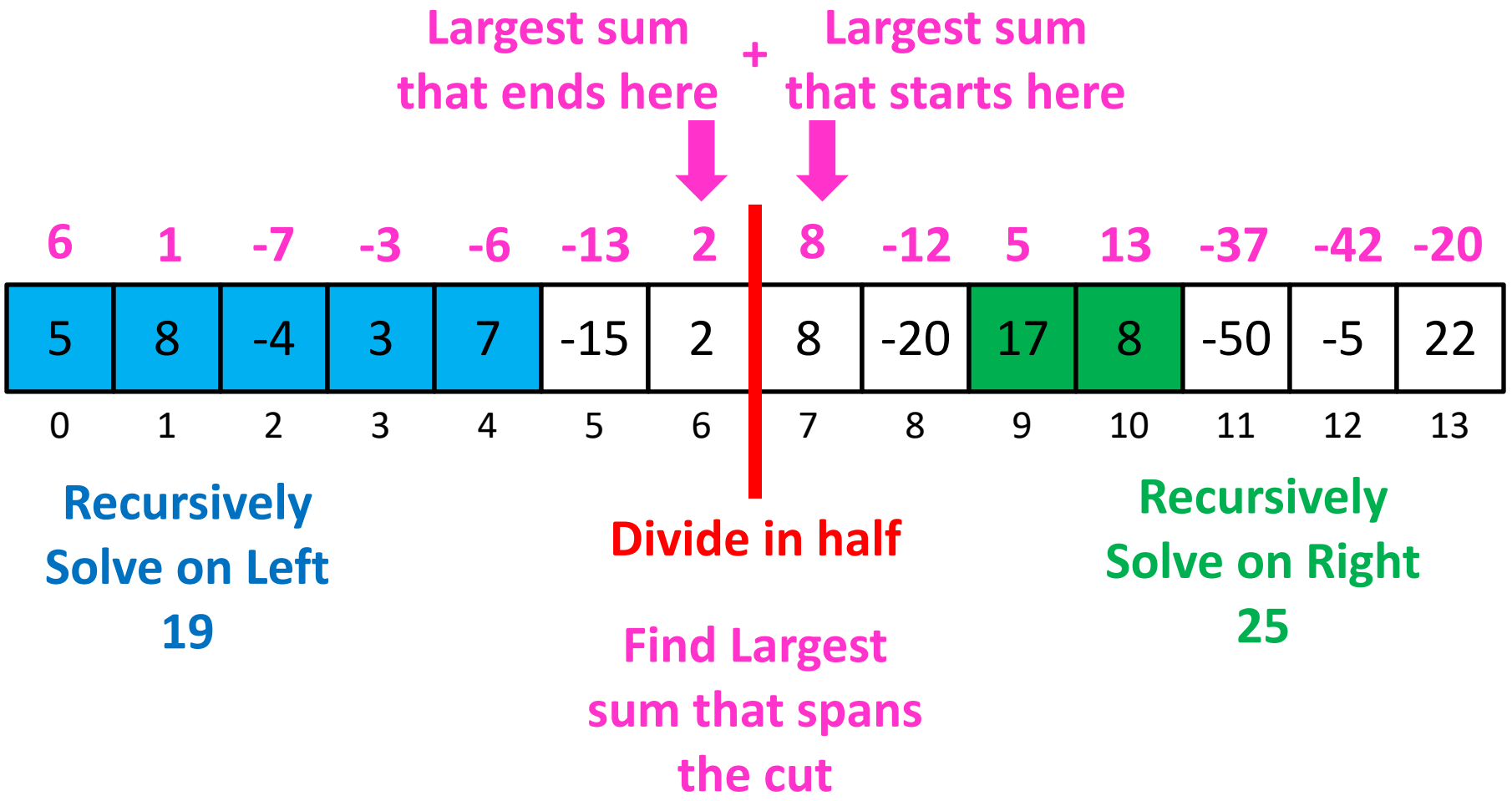
The maximum-sum subarray of a given array of integers A is the interval $[a, b]$ such that the sum of all values in the array between a and b inclusive is maximal.

Given an array of n integers (may include both positive and negative values), give a $O(n \log n)$ algorithm for finding the maximum-sum subarray.

Divide and Conquer $\Theta(n \log n)$

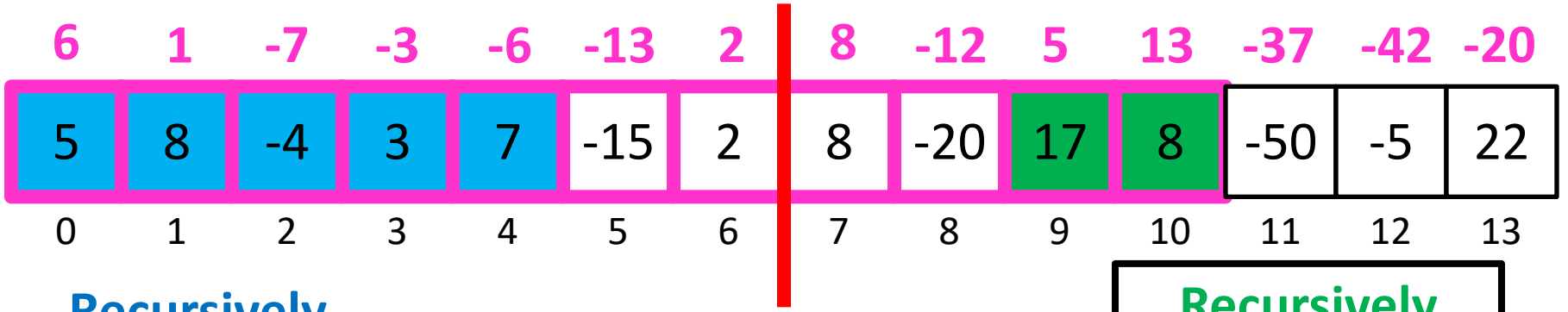


Divide and Conquer $\Theta(n \log n)$



Divide and Conquer $\Theta(n \log n)$

Return the Max of
Left, Right, Center



Recursively Solve on Left
19

Divide in half

Find Largest sum that spans the cut
19

Recursively Solve on Right
25

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$