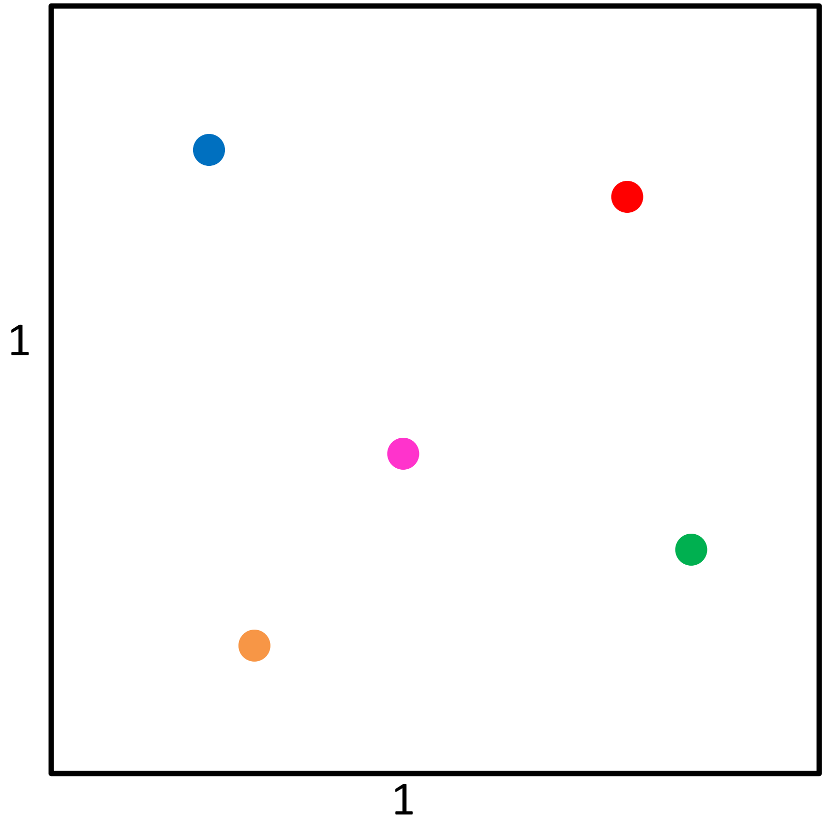
Warm up

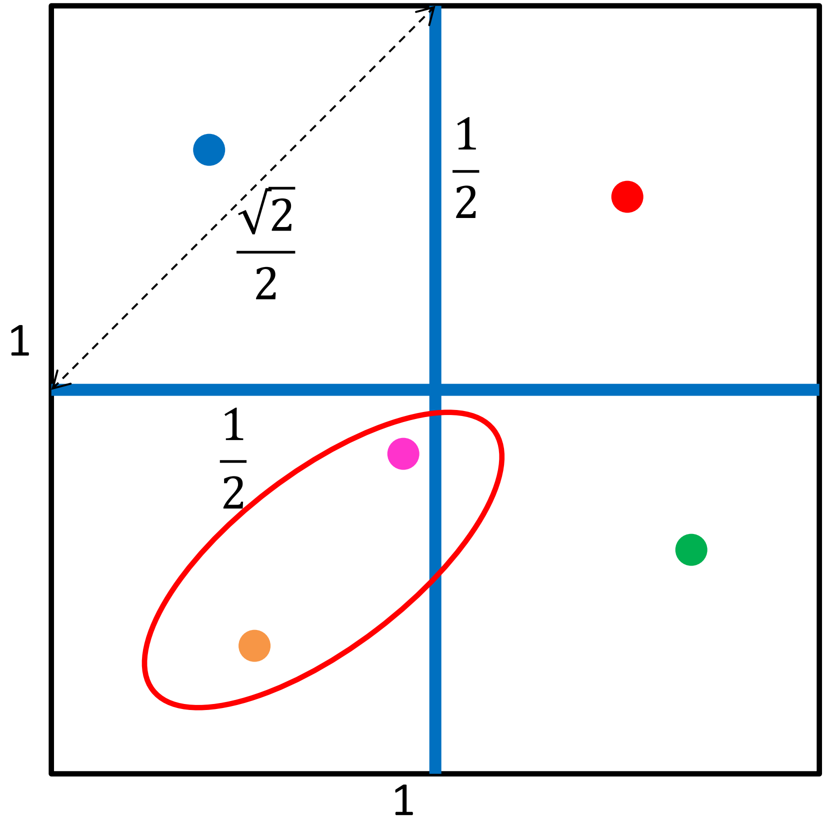
Given any 5 points on the unit square, show there's always a pair distance $\leq \frac{\sqrt{2}}{2}$ apart



If points p_1, p_2 in same quadrant, then $\delta(p_1, p_2) \leq \frac{\sqrt{2}}{2}$

Given 5 points, two must share the same quadrant

Pigeonhole Principle!



Today's Keywords

- Karatsuba (one last time!)
- Solving recurrences
- Cookbook Method
- Master Theorem
- Substitution Method

CLRS Readings

- Chapter 4

Homeworks

Friday, January 31 at 11pm

- HW1 due ~~Thursday, January 30 at 11pm~~
  - Written (use Latex!) – Submit BOTH pdf and zip!
  - Asymptotic notation
  - Recurrences
  - Divide and Conquer
- HW2 out tomorrow, due Thursday, February 6 at 11pm
  - Written (use Latex!)
  - Divide and Conquer
  - Master Theorem

Recurrence Solving Techniques

Tree

? ✓ Guess/Check (induction)

"Cookbook"

Substitution

Induction (review)

Goal: $\forall k \in \mathbb{N}, P(k)$ holds

Base case(s): $P(1)$ holds

Technically, called *strong induction*

Hypothesis: $\forall x \leq x_0, P(x)$ holds

Inductive step: show $P(1), \ldots, P(x_0) \Rightarrow P(x_0 + 1)$

Guess and Check Intuition

- **Show:** $T(n) \in O(g(n))$
- **Consider:** $g_*(n) = c \cdot g(n)$ for some constant c, i.e. pick $g_*(n) \in O(g(n))$
- **Goal:** show $\exists n_0$ such that $\forall n > n_0, T(n) \le g_*(n)$
  - (definition of big-O)
- **Technique:** Induction
  - Base cases:
    - show $T(1) \le g_*(1), T(2) \le g_*(2)$, … for a small number of cases (may need additional base cases)
  - Hypothesis:
    - $\forall n \le x_0, T(n) \le g_*(n)$
  - Inductive step:
    - Show $T(x_0 + 1) \le g_*(x_0 + 1)$

Need to ensure that in inductive step, can either appeal to a <u>base case</u> or to the <u>inductive hypothesis</u>

Karatsuba Algorithm

$$\begin{array}{r} a \quad b \\ \times \; c \quad d \\ \hline \end{array}$$

1. Recursively compute: $ac, bd, (a+b)(c+d)$
2. $(ad+bc) = (a+b)(c+d) - ac - bd$
3. Return $10^n(ac) + 10^{\frac{n}{2}}(ad+bc) + bd$

<u>Pseudo-code</u>

1. $x \leftarrow \text{ Karatsuba}(a, c)$
2. $y \leftarrow \text{ Karatsuba}(b, d)$
3. $z \leftarrow \text{ Karatsuba}(a+b, c+d) - x - y$
4. Return $10^n x \; + 10^{n/2} z + y$

$$\boxed{T(n) = 3T\left(\frac{n}{2}\right) + 8n}$$

Karatsuba

3. Use asymptotic notation to simplify

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$$T(n) = 8n \sum_{i=0}^{\log_2 n} \left({}^{3}/_{2}\right)^{i}$$

$$T(n) = 8n \frac{\left({}^{3}/_{2}\right)^{\log_2 n+1} - 1}{{}^{3}/_{2} - 1}$$

Math, math, and more math…(on board, see lecture supplement)

$$T(n) = 24\left(n^{\log_2 3}\right) - 16n \;= \Theta\left(n^{\log_2 3}\right)$$
$$\approx \Theta\left(n^{1.585}\right)$$

Karatsuba Guess and Check

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

$g_*(n) = 24n^{\log_2 3} - 16n$

Goal: $T(n) \leq 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base cases: by inspection, holds for small n (at home)

Hypothesis: $\forall n \leq x_0, T(n) \leq 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

Karatsuba Guess and Check

$T(n) = 3T\left(\frac{n}{2}\right) + 8n$

Hyp: $\forall n \le x_0 \quad T(n) \le \underbrace{24n^{\log_2 3} - 16n}$

$2^{\log_2 3} = 3$

Inductive Step: show $T(x_0+1) \le 24(x_0+1)^{\log_2 3} - 16(x_0+1)$

$$T(x_0+1) = 3\underbrace{T\left(\overset{\le x_0}{\boxed{\frac{x_0+1}{2}}}\right)} + 8(x_0+1)$$

$$\le 3\left(24\left(\frac{x_0+1}{2}\right)^{\log_2 3} - 16\left(\frac{x_0+1}{2}\right)\right) + 8(x_0+1)$$

$$= 3\left(\frac{24}{3}(x_0+1)^{\log_2 3} - 8(x_0+1)\right) + 8(x_0+1)$$

$$= 24(x_0+1)^{\log_2 3} - 24(x_0+1) + 8(x_0+1)$$

$$= 24(x_0+1)^{\log_2 3} - 16(x_0+1)$$

Therefore $T(n) \in O(n^{\log_2 3})$

What if we leave out the $-16n$?

$$T(n) = 3T\left(\frac{n}{2}\right) + 8n$$

Goal: $T(n) \leq 24n^{\log_2 3} - 16n = O(n^{\log_2 3})$

Base cases: by inspection, holds for small n (at home)

Hypothesis: $\forall n \leq x_0, T(n) \leq 24n^{\log_2 3} - 16n$

Inductive step: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} - 16(x_0 + 1)$

What we wanted: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3}$ Induction failed!

What we got: $T(x_0 + 1) \leq 24(x_0 + 1)^{\log_2 3} + 8(x_0 + 1)$

"Bad Mergesort" Guess and Check

$$T(n) = 2\,T\left(\frac{n}{2}\right) + 209n$$

Goal: $T(n) \leq 209n\log_2 n \;\; = O(n\log_2 n)$

Base cases: $T(1) = 0$
$T(2) = 518 \leq 209 \cdot 2\log_2 2$
… up to some small k

Hypothesis: $\forall n \leq x_0, T(n) \leq 209n\log_2 n$

Inductive step: $T(x_0 + 1) \leq 209(x_0 + 1)\log_2(x_0 + 1)$

Prove this on your own

Recurrence Solving Techniques

Tree

? ✓ Guess/Check

"Cookbook"

Substitution

Observation

- **Divide:** $D(n)$ time
- **Conquer:** recurse on small problems, size s
- **Combine:** $C(n)$ time
- **Recurrence:**

$$T(n) = D(n) + \sum T(s) + C(n)$$

- Many D&C recurrences are of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n), \qquad \text{where } f(n) = D(n) + C(n)$$

Remember…

- Better Attendance: $T(n) = T\left(\frac{n}{2}\right) + 2$
- MergeSort: $T(n) = 2\,T\left(\frac{n}{2}\right) + n$
- D&C Multiplication: $T(n) = 4T\left(\frac{n}{2}\right) + 5n$
- Karatsuba: $T(n) = 3T\left(\frac{n}{2}\right) + 8n$

General

$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right)$$

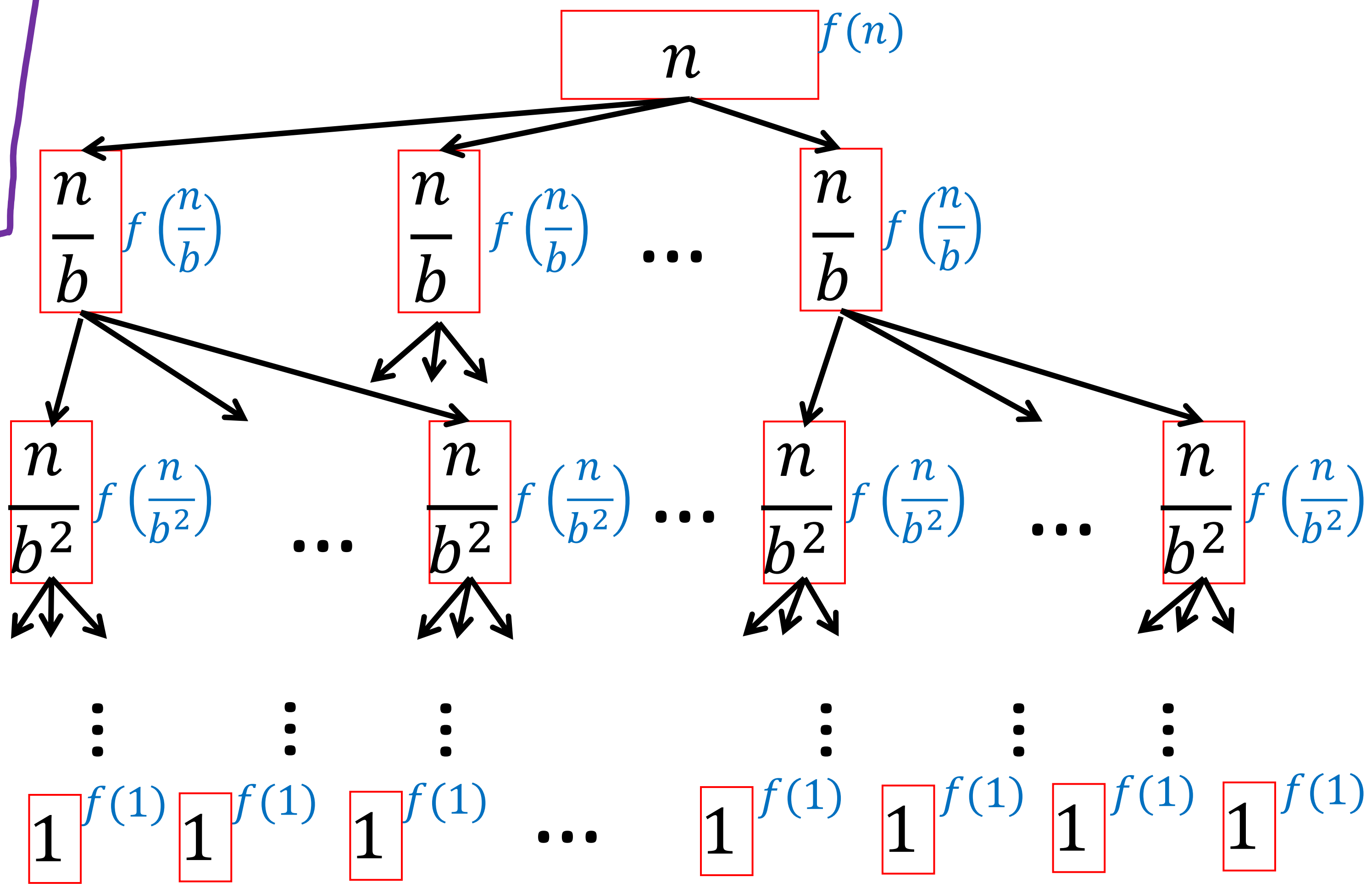
$a^{\log_b n}$

$= \left(b^{\log_b a}\right)^{\log_b n}$

$= \left(b^{\log_b n}\right)^{\log_b a}$

$= n^{\log_b a}$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



$f(n)$

$af\left(\frac{n}{b}\right)$

$a^2 f\left(\frac{n}{b^2}\right)$

$\vdots$

$a^{\log_b n} f(1)$

3 Cases

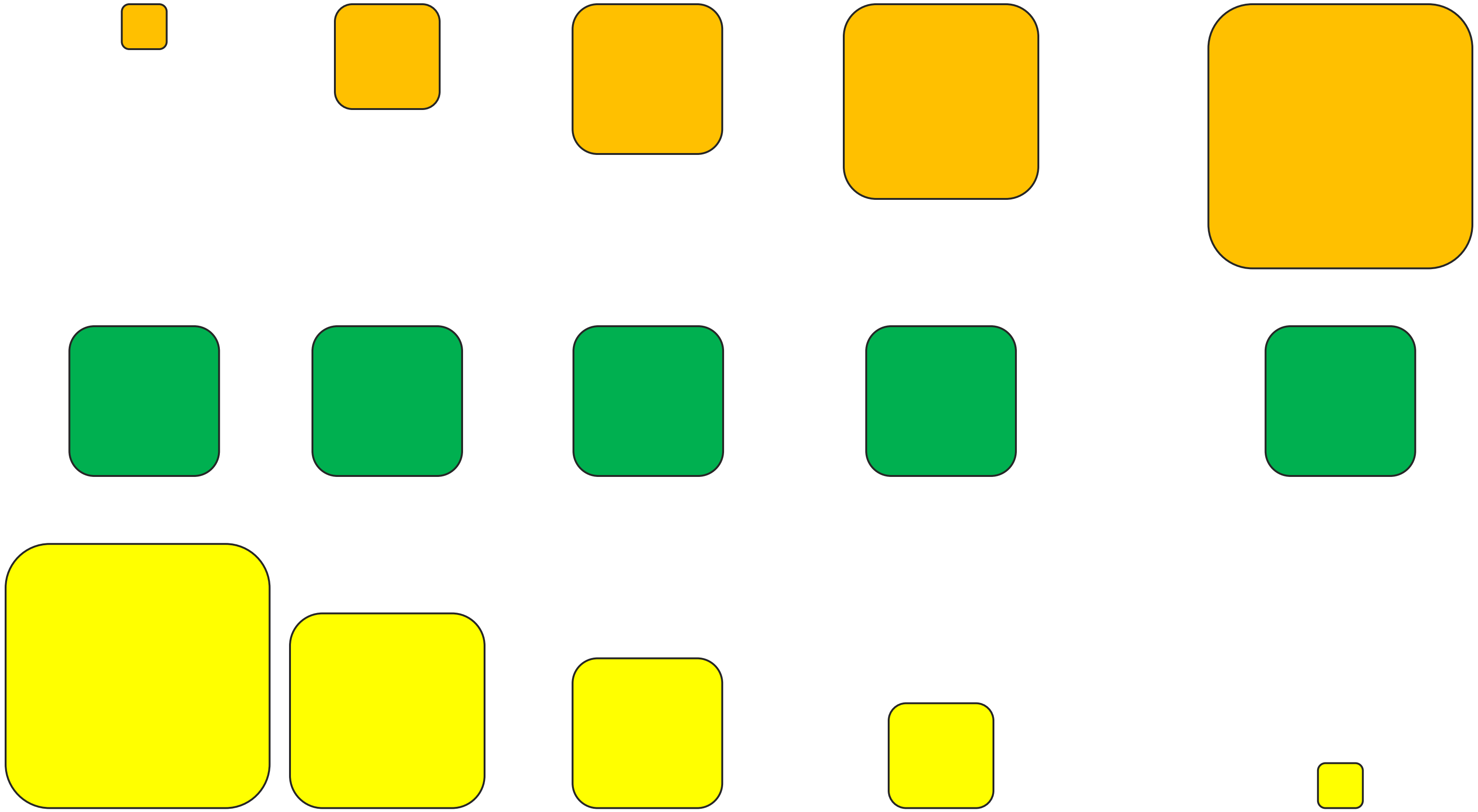
$L = \log_b n$

$$T(n) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + a^3 f\left(\frac{n}{b^3}\right) + \cdots + a^L f\left(\frac{n}{b^L}\right)$$

Case 1:
Most work happens at the leaves

Case 2:
Work happens consistently throughout

Case 3:
Most work happens at top of tree



Master Theorem

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Case 1: if $f(n) \in O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$,

then $T(n) \in \Theta\left(n^{\log_b a}\right)$

Case 2: if $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a} \log n)$

Case 3: if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$,

and if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$

and all sufficiently large n,

then $T(n) \in \Theta(f(n))$

Proof of Case 1

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right),$$

$$f(n) \in O\left(n^{\log_b a - \varepsilon}\right) \Rightarrow f(n) \leq c \cdot n^{\log_b a - \varepsilon}$$

Insert math here…

Proof of Case 1

$T(n) = aT(\frac{n}{b}) + f(n) = \sum_{i=0}^{\log_b n} a^i f(\frac{n}{b^i})$

$L = \log_b n$

Assume: $f(n) \in O(n^{\log_b a - \varepsilon})$

$\exists c, n_0 > 0 \quad \forall n > n_0 \quad f(n) \leq c\, n^{\log_b a - \varepsilon}$

$T(n) = f(n) + af(\frac{n}{b}) + a^2 f(\frac{n}{b^2}) + \dots + a^L f(1)$

$\leq c\left(n^{\log_b a-\varepsilon} + a\left(\frac{n}{b}\right)^{\log_b a-\varepsilon} + a^2\left(\frac{n}{b^2}\right)^{\log_b a-\varepsilon} + \dots + a^{L-1}\left(\frac{n}{b^{L-1}}\right)^{\log_b a-\varepsilon}\right) + a^L f(1)$

$= c\,n^{\log_b a-\varepsilon}\left(1 + \frac{a}{b^{\log_b a-\varepsilon}} + \frac{a^2}{b^{2(\log_b a-\varepsilon)}} + \dots + \frac{a^{L-1}}{b^{(L-1)(\log_b a-\varepsilon)}}\right) + a^L f(1)$

$= c\,n^{\log_b a-\varepsilon}\left(1 + b^{\varepsilon} + b^{2\varepsilon} + \dots + b^{(L-1)\varepsilon}\right) + a^L f(1)$

$= c\,n^{\log_b a-\varepsilon}\left(\frac{b^{\varepsilon L} - 1}{b^{\varepsilon} - 1}\right) + a^L f(1) = c\,n^{\log_b a-\varepsilon}\left(\frac{b^{\varepsilon \log_b n} - 1}{b^{\varepsilon} - 1}\right) + a^L f(1)$

$= c\,n^{\log_b a-\varepsilon}\left(\frac{n^{\varepsilon} - 1}{b^{\varepsilon} - 1}\right) + a^L f(1) = c\,n^{\log_b a-\varepsilon}\left((n^{\varepsilon}-1)\underbrace{\frac{1}{b^{\varepsilon}-1}}_{c_2}\right) + a^{\log_b n}\underbrace{f(1)}_{c_3}$

$b^{\log_b a-\varepsilon}$

$= b^{\log_b a} \cdot b^{-\varepsilon}$

$= \frac{a}{b^{\varepsilon}}$

$b^{2(\log_b a-\varepsilon)}$

$= \frac{a^2}{b^{2\varepsilon}}$

Proof of Case 1

$$T(n) \le \underline{c_2 \cdot c}\, n^{\log_b a - \varepsilon}\left(n^{\varepsilon} - 1\right) + c_3 n^{\log_b a}$$

$$= c_4 n^{\log_b a} - c_4 n^{\log_b a - \varepsilon} + c_3 n^{\log_b a}$$

$$= c_5 n^{\log_b a} - c_4 n^{\log_b a - \varepsilon}$$

$$\le c_5 n^{\log_b a} \in O\left(n^{\log_b a}\right)$$

Conclusion: $T(n) = O(n^{\log_b a})$